M. W. Kalinowski
(Higher Vocational State School in Chełm, Poland)

Comment on a Quintessence Particle Mass in the Kaluza–Klein Theory and Properties of its Field

Abstract. We calculate a mass of a quintessence particle of order 10^{-5} eV and we find several solutions for quintessence field equation. We consider also a quintessence speed of sound in several schemes and quintessence fluctuations.

In this paper we consider several consequences of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. We consider a value of a mass of quintessence particle, several interesting relations among energy scales, radiation density in the second de Sitter phase. We find a spatial dependence of a quintessence field (and an effective gravitational constant G_{eff}) in a case of spherical-statical symmetry, cylindrical-statical symmetry, flat-static symmetry. We find a time dependences of a quintessence field (with no spatial dependence). We get a solution for a quintessence field (a travelling wave) and two-dimensional wave solution applying those solutions for G_{eff}. We propose some kind of statistical approach to our results. We calculate a speed of sound in a quintessence. We consider also fluctuations of a quintessence caused by gravitational waves perturbations.

Let us consider a value of mass of a quintessence particle (a scalar particle) (see Ref. [1]) obtained from Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory (see Ref. [2]):

$$m_0 = \sqrt{\frac{n}{2}}\left(\frac{n}{n+2}\right)^{n/4} |\gamma|^{1/2}\left(\frac{|\gamma|}{\beta}\right)^{n/4}. \tag{1}$$

(see Eq. (79) from Ref. [1]).

The value of this mass has been obtained by this particle during the second de Sitter phase. Moreover during our contemporary epoch it is the same. The parameters n, γ and β are defined in Refs [1], [2]. During the second de Sitter phase a cosmological constant has been calculated in Ref. [2] and one finds

$$\lambda_{c0} = 6H_1^2 = \frac{2|\gamma|^{(n+2)/2} n^{n/2}}{\beta^{n/2}(n+2)^{(n+2)/2}}\,. \tag{2}$$

The value of a cosmological constant remains the same during our contemporary epoch. H_1 is a Hubble constant during the second de Sitter phase (see Eq. (8) of Ref. [1]).

Thus for our contemporary epoch one gets

$$m_0 = \frac{1}{2}\sqrt{n(n+2)}\,\sqrt{\lambda_{c0}}\,. \tag{3}$$

In this way we can connect the value of a cosmological constant of our contemporary epoch to the value of a mass of a quintessence particle. From recent observational data we get

$$\lambda_{c0} = \Lambda = 10^{-52} \frac{1}{\mathrm{m}^2} . \tag{4}$$

Moreover in order to get a correct dimension for a mass we should add a factor with $\hbar$ and c (now we abandon the system of units with $\hbar = c = 1$).

One gets

$$m_0 = \frac{1}{2} \frac{\hbar}{c} \sqrt{n(n+2)} \sqrt{\lambda_{c0}} . \tag{3a}$$

Using the value of λ_{c0} one finally gets

$$m_0 \simeq \sqrt{n(n+2)} \cdot 0.17 \cdot 10^{-39}\,\mathrm{g} \tag{5}$$

or

$$m_0 \simeq \sqrt{n(n+2)} \cdot 0.95 \cdot 10^{-5}\,\mathrm{eV}. \tag{5a}$$

For example, if we take $n = 14 (= \dim G2)$, one finally gets

$$m_0 \simeq 14.2 \cdot 10^{-5}\,\mathrm{eV}. \tag{6}$$

This value is bigger than that considered by different authors. Moreover, still sufficiently small. The particle interacts only gravitationally and because of this it is undetectable by using known experimental methods. Taking a density of dark energy as 0.7 of a critical density,

$$\rho_c = 1.88 h^2 \cdot 10^{-29} \frac{\mathrm{g}}{\mathrm{cm}^3} , \tag{7}$$

one gets a number of quintessence particles per unit volume

$$\overline{n} = \frac{h^2}{\sqrt{n(n+2)}} \cdot 1.31 \cdot 10^{10} \frac{1}{\mathrm{cm}^3} \tag{8}$$

where h is a dimensionless Hubble constant $0.7 < h < 1$. Taking $n = 14$ and $h = 0.7$ one finally gets

$$\overline{n} = 4 \cdot 10^8 \frac{1}{\mathrm{cm}^3} \tag{8a}$$

which is many orders of magnitude smaller than Loschmidt number. Thus a gas of quintessence particles is not so dense from the point of view of our earth conditions. However, if this number of particles per unit volume is considered in a container of size 200 Mpc, the gas can be considered as extremely dense.

In order to settle—is this gas dense or not—we should calculate a mean scattering length. The scattering cross-section for a quintessence particle

$$\sigma = \frac{1}{\lambda_{c0}} = 10^{52}\,\mathrm{m}^2. \tag{$*$}$$

A mean scattering length

$$l = \frac{1}{\sigma n} \tag{$**$}$$

where n is a number of quintessence particles per unit volume (Eq. (8a)).

One gets

$$l = 10^{-60}\,\mathrm{m}. \tag{$***$}$$

It means that a gas of quintessence particles is extremely dense (if we apply the Knudsen criterion—a gas is dense if $l \ll L$, where L is the size of the container) even in the Solar System.

Let us consider the Eq. (111) from Ref. [3] which connects several scales of energy and gives an account of the smallness of gravitational interactions in our theory. We rewrite this equation in the form

$$\left(\frac{m_{\mathrm{pl}}}{m_{\mathrm{EW}}}\right)\left(\frac{m_{\tilde{A}}}{m_{\mathrm{EW}}}\right)^{n_1} = \left(\frac{n|\gamma|}{(n+2)\beta}\right)^{(n+2)(n_1+2)/2} \tag{9}$$

where m_{EW} is an electro-weak interactions energy scale. Taking

$$\begin{aligned} &m_{\tilde{A}} = m_{\mathrm{EW}} \\ &n_1 = 2 \ (M = S^2) \\ &n = 14 \ (H = G2) \end{aligned}$$

one gets

$$\frac{m_{\mathrm{pl}}}{m_{\mathrm{EW}}} = \left(\frac{7}{8}\,\kappa\right)^{24} \tag{10}$$

where

$$\kappa = \frac{|\gamma|}{\beta} \tag{11}$$

and eventually one gets

$$\kappa = \left(\frac{m_{\mathrm{pl}}}{m_{\mathrm{EW}}}\right)^{1/24} \cdot \frac{8}{7}\,. \tag{12}$$

Taking $m_{\mathrm{EW}} \simeq 80\,\mathrm{GeV}$ and $m_{\mathrm{pl}} \simeq 2.4 \cdot 10^{18}\,\mathrm{GeV}$ one gets

$$\kappa \simeq 6.19 \tag{13}$$

which is very reasonable for it establishes a relation between two cosmological terms γ and β as of the same order. Simultaneously this is a consistency condition for our model with energy scales (the 6-dimensional Planck's mass is equal to $m_{\rm EW}$). In this way we can calculate a mass of a quintessence particle for our contemporary epoch from Eq. (1)

$$m_0 = 2 \cdot 10^{15}\,\mathrm{eV} \cdot |\underline{\widetilde{P}}|^{1/2}. \tag{14}$$

This gives us an estimation for $|\underline{\widetilde{P}}|$ and $\widetilde{R}(\widetilde{\Gamma})$:

$$\widetilde{R}(\widetilde{\Gamma}) = \frac{1}{\kappa}\left(\frac{m_{\rm EW}}{m_{\rm pl}\alpha_{\rm em}}\right)^2 |\underline{\widetilde{P}}| \tag{15}$$

where $\alpha_{\rm em} = \frac{1}{137}$ is a fine coupling constant. Using Eq. (13) and values of $m_{\rm pl}$ and $m_{\rm EW}$ one gets

$$\widetilde{R}(\widetilde{\Gamma}) = 24.75 \cdot 10^{-31} |\underline{\widetilde{P}}|. \tag{16}$$

From Eq. (14) and Eq. (6) one gets

$$|\underline{\widetilde{P}}| = 5 \cdot 10^{-39} \tag{17}$$

and

$$\widetilde{R}(\widetilde{\Gamma}) \cong 1.2 \cdot 10^{-68}. \tag{18}$$

Moreover in our simplified theory we have

$$\widetilde{R}(\widetilde{\Gamma}) = \frac{2(2\mu^3 + 7\mu^2 + 5\mu + 20)}{(\mu^2+4)^2} \tag{19}$$

and μ should be very close to the root of the polynomial

$$W(\mu) = 2\mu^3 + 7\mu^2 + 5\mu + 20. \tag{20}$$

From (18) and (19) one gets

$$2\mu^3 + 7\mu^2 + 5\mu + 20 \cong 1.3 \cdot 10^{-66}. \tag{21}$$

Thus μ is very close to the 70-digit approximation of the root of the polynomial (20) $(W(\overline{\mu}) = 0.1 \cdot 10^{-67})$. Due to this we can control the cosmological terms.

In the case of $|\underline{\widetilde{P}}|$ we have the formula (28) from Ref. [1] and one gets

$$\underline{\widetilde{P}}(\zeta_0 + \varepsilon) \cong \underline{\widetilde{P}}(\zeta_0) + \frac{d\underline{\widetilde{P}}}{d\zeta}(\zeta_0)\varepsilon \tag{22}$$

$$\underline{\widetilde{P}}(\zeta_0 = \pm 1.38\dots) = 0 \tag{23}$$

$$\left|\frac{d\underline{\widetilde{P}}}{d\zeta}(|\zeta_0| = 1.38\dots)\right| = 25. \tag{24}$$

Thus one gets from (17) and (23–24)

$$\varepsilon \simeq 2 \cdot 10^{-40}. \tag{25}$$

Thus we need an approximation of ζ_0 up 40-digit arithmetics. We see that cosmological terms coming from the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory are very small, but not zero, and that they are easily controllable by μ and ζ parameters.

Let us consider a self-interaction potential for a quintessence field for our contemporary epoch (which is the same as for the second de Sitter phase). One gets from cosmological terms $\lambda_{c0}(\Psi)$

$$\lambda_{c0}(\Psi) = -\frac{1}{2}\left(\beta e^{2\Psi} - |\gamma|\right) e^{n\Psi} \tag{26}$$

$$\begin{aligned} U(q_0) = {} & -\frac{|\gamma|}{2(n+2)}\left(\frac{n|\gamma|}{(n+2)\beta}\right)^{\frac{n}{2}} \exp\left(\frac{n m_{\mathrm{pl}}}{2\sqrt{2\pi|\overline{M}|}} q_0\right) \\ & \times \left(n \exp\left(\frac{m_{\mathrm{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0\right) - (n+2)\right) \end{aligned} \tag{27}$$

or

$$U(q_0) = -\frac{1}{4}\lambda_{c0} \exp\left(\frac{n m_{\mathrm{pl}}}{2\sqrt{2\pi|\overline{M}|}} q_0\right)\left(n \exp\left(\frac{m_{\mathrm{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0\right) - (n+2)\right) \tag{27a}$$

where λ_{c0} is a cosmological constant for our contemporary epoch and

$$\Psi = \Psi_0 + \frac{m_{\mathrm{pl}}}{2\sqrt{2\pi|\overline{M}|}} q_0 = \Psi_0 + \overline{\beta} q_0 = \Psi_0 + \varphi. \tag{28}$$

For λ_{c0} is very small $(10^{-52}\frac{1}{\mathrm{m}^2})$, this interaction is very small. For $n = 14 (= \dim H = \dim G2)$ one gets

$$U(q_0) = -\frac{1}{2}\lambda_{c0} \exp\left(\frac{7 m_{\mathrm{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0\right)\left(7 \exp\left(\frac{m_{\mathrm{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0\right) - 8\right). \tag{29}$$

Even in the potential $U(q_0)$ we have exponential terms, the strength of the interactions is negligible for small value of q_0. The interesting point in our theory is the effective gravitational constant (depending on a scalar field Ψ). Let us describe it by a quintessence field q_0. One gets

$$G_{\mathrm{eff}} = G_0 e^{-(n+2)\Psi}. \tag{30}$$

After some calculations one finds

$$G_{\text{eff}} = G_0 \left(\frac{(n+2)\beta}{n|\gamma|} \right)^{n+2} \exp\left(-\frac{n+2}{2} \frac{m_{\text{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0 \right) \tag{31}$$

or

$$G_{\text{eff}} = \frac{G_0}{\lambda_{c0}^2} \cdot \frac{4(n+2)^2\beta^2}{n^2} \exp\left(-\frac{n+2}{2} \frac{m_{\text{pl}}}{\sqrt{2\pi|\overline{M}|}} q_0 \right). \tag{31a}$$

It is easy to see that the rôle of a Newton's constant G_N is played by

$$G_N = \frac{G_0}{\lambda_{c0}^2} \left(\frac{2(n+2)\beta}{n} \right)^2. \tag{32}$$

This is the reason that we put a constant G_0 in the formula (30).

One gets

$$G_0 = G_N \lambda_{c0}^2 \left(\frac{n}{2(n+2)\beta} \right)^2. \tag{33}$$

For

$$\beta = \alpha_s^2 m_{\text{pl}}^2 \widetilde{R}(\widetilde{\Gamma}) = \alpha_s^2 \frac{\widetilde{R}(\widetilde{\Gamma})}{l_{\text{pl}}^2} \tag{34}$$

one gets

$$G_N = \frac{G_0 \alpha_s^4}{(\lambda_{c0} l_{\text{pl}}^2)^2} \left(\frac{2(n+2)\widetilde{R}(\widetilde{\Gamma})}{n} \right)^2 \tag{35}$$

or

$$G_0 = G_N \left(\frac{n}{2(n+2)\widetilde{R}(\widetilde{\Gamma})} \right)^2 \left(\frac{\lambda_{c0} l_{\text{pl}}^2}{\alpha_s^2} \right)^2. \tag{36}$$

Let us connect to G_0 a new Planck's mass m_{pl}^0. In terms of this mass and ordinary m_{pl} mass one gets

$$m_{\text{pl}}^0 = m_{\text{pl}} \frac{2(n+2)\widetilde{R}(\widetilde{\Gamma})}{n} \frac{\alpha_s^2}{\lambda_{c0} l_{\text{pl}}^2} \tag{37}$$

or

$$m_{\text{pl}} = m_{\text{pl}}^0 \frac{n}{2(n+2)\widetilde{R}(\widetilde{\Gamma})} \frac{\lambda_{c0} l_{\text{pl}}^2}{\alpha_s^2}. \tag{38}$$

If we take our simplified model with $\widetilde{R}(\widetilde{\Gamma})$ given by Eq. (19)

$$\alpha_s^2 = \alpha_{\text{em}} = \tfrac{1}{137}, \ n = 14,$$

we get

$$m_{\mathrm{pl}}^{0} = m_{\mathrm{pl}} \frac{32(2\mu^3 + 7\mu^2 + 5\mu + 20)}{7(\mu^2 + 4)^2} \frac{\alpha_{\mathrm{em}}}{\lambda_{c0} l_{\mathrm{pl}}^2} \tag{39}$$

or

$$G_0 = G_N \left(\frac{7(\mu^2 + 4)^2}{32(2\mu^3 + 7\mu^2 + 5\mu + 20)} \right)^2 \left(\frac{\lambda_{c0} l_{\mathrm{pl}}^2}{\alpha_{\mathrm{em}}} \right)^2 . \tag{40}$$

Moreover we can connect unknown constant G_0 (m_{pl}^0) with an energy scale of electroweak interactions m_{EW} in a way suggested in Ref. [3] and developed here in this paper.

It is enough to use the formula (10) and we get

$$m_{\mathrm{pl}}^{0} = m_{\mathrm{EW}} \left(\frac{7\kappa}{8} \right)^{24} \frac{32(2\mu^3 + 7\mu^2 + 5\mu + 20)}{7(\mu^2 + 4)^2} \frac{\alpha_{\mathrm{em}}}{\lambda_{c0} l_{\mathrm{pl}}^2} . \tag{41}$$

If we take for λ_{c0} the known value of the cosmological constant (Eq. (4)) for $l_{\mathrm{pl}} = 4 \cdot 10^{-32}$ cm and $\kappa = 6.19$ we get

$$m_{\mathrm{pl}}^{0} = 8.12 \cdot 10^{132} \cdot \frac{2\mu^3 + 7\mu^2 + 5\mu + 20}{(\mu^2 + 4)^2} m_{\mathrm{EW}} . \tag{42}$$

Moreover we see that μ must be very close to the root of the polynomial $2\mu^3 + 7\mu^2 + 5\mu + 20$ (see Ref. [3]) and we get

$$m_{\mathrm{pl}}^{0} = 2 \cdot 10^{130} (2\mu^3 + 7\mu^2 + 5\mu + 20) m_{\mathrm{EW}} . \tag{43}$$

If we take 70-digit approximation of the root of the polynomial (20) and considered value of $W(\mu)$ (Eq. (18)) obtained here, we get

$$m_{\mathrm{pl}}^{0} = 4.6 \cdot 10^{64} m_{\mathrm{EW}} \tag{44}$$

or

$$m_{\mathrm{pl}}^{0} = 4.6 \cdot 10^{64} \frac{m_{\mathrm{EW}}}{m_{\mathrm{pl}}} m_{\mathrm{pl}} = 1.5 \cdot 10^{48} m_{\mathrm{pl}} = 3.2 \cdot 10^{43} \mathrm{g} = 1.6 \cdot 10^{10} M_{\odot} \tag{45}$$

where $M_{\odot} = 1.99 \cdot 10^{33}$ g is a Solar mass. (This is a mass of our Galaxy.)

Moreover the present critical density of matter in the Universe is

$$\rho_{c,0} = 2.775 h^{-1} \times 10^{11} \frac{M_{\odot}}{(h^{-1}\,\mathrm{Mpc})^3} . \tag{46}$$

So we can find a volume containing m_{pl}^0. One gets

$$V_0 = \frac{m_{\mathrm{pl}}^{0}}{\rho_{c,0}} = 5 \cdot 10^{-3} h^2 (\mathrm{Mpc})^3 . \tag{47}$$

Thus the size of this volume is of order

$$L_0 \simeq \sqrt[3]{h^2}\,0.17\,\mathrm{Mpc}. \tag{48}$$

Taking under consideration the fact that our calculations are quite rough (for example κ could be bigger a little, or $W(\widetilde{\mu})$ a little bigger), we come to the conclusion that m_{pl}^0 is of order of the mass of our visible Universe (a volume of size of 10^4 Mpc). In this way it seems natural to suppose that

$$m_{\mathrm{pl}}^0 = M_U \tag{49}$$

where M_U is the total mass of the visible Universe. However, this intriguing conjecture demands many investigations.

Moreover it is in a real spirit of Dirac's large number hypothesis and can give a link between cosmology and fundamental interactions theory. Thus if we suppose (49) we get

$$G_0 = G_U = \frac{1}{M_U^2}\,. \tag{50}$$

Let us come back to the equation for an effective gravitational constant (Eq. (32)):

$$G_{\mathrm{eff}} = G_N \exp\left(-\frac{n+2}{2}\,\frac{m_{\mathrm{pl}}}{\sqrt{2\pi|\overline{M}|}}\,q_0\right). \tag{51}$$

q_0—a quintessence scalar field—possesses a mass and because of this it has a finite range with Youkave type behaviour

$$q_0 = \frac{\alpha}{R}\exp\left(-\frac{R}{r_0}\right) \tag{52}$$

where

$$r_0 = \frac{2}{\sqrt{n(n+2)}\sqrt{\lambda_{c0}}} \quad \text{and } \alpha \text{ is a positive constant} \tag{53}$$

or taking for $n = 14$ and for λ_{c0} Eq. (4),

$$r_0 = 3\cdot 10^{23}\,\mathrm{m} = 10\,\mathrm{Mpc}. \tag{54}$$

The formula (52) is valid for $R \simeq r_0$. Thus we cannot observe scalar gravitational radiation from closed binary sources and the quadrupole radiation formula is satisfied for a gravitational field.

Moreover for $R < r_0$ we should take under consideration a full self-interaction potential of a field q_0 (Eq. (29)). Moreover, because of the constant λ_{c0} in front of the formula (29) this is negligible. Because of this we can repeat some considerations from

the first point of Ref. [2] coming back to the field Ψ and consider different sources of mass for this field due to interaction with the matter. In this way in both cases

$$G_{\text{eff}} = G_N\,. \tag{55}$$

However, we can expect some small effects on short distances.

Let us notice that in the previous approximation we consider a weak field, it means $|q_0| \ll 1$. This is different from small field. A small field considered below is such that q_0 is small in a usual sense. It is negative. Moreover it can be big in the sense of the absolute value.

Let us calculate a radiation density in the moment when the second de Sitter phase starts. It means the radiation released after the phase transition. One gets

$$\rho_r = \lambda_{c1}(\Psi_1) - \lambda_{c0}(\Psi_0) = \lambda_{c0}(\Psi_0)\left(\frac{H_0^2}{H_1^2} - 1\right). \tag{56}$$

Using Eqs (1a) and (8a) from Ref. [2] one gets

$$\begin{aligned}\rho_r &= \lambda_{c0}(\Psi_0) \\ &\times \Bigg(\frac{n^2|\underline{\widetilde{P}}|^2 + |\underline{\widetilde{P}}|\sqrt{n^2|\underline{\widetilde{P}}|^2 + 4(n^2-4)A\widetilde{R}(\widetilde{\Gamma})} + 4A\widetilde{R}(\widetilde{\Gamma})(n+2)}{2^{(n+2)/2}|\underline{\widetilde{P}}|^{(n+2)/2}n^{n/2}} \\ &\quad\times \left(n|\underline{\widetilde{P}}| + \sqrt{n^2|\underline{\widetilde{P}}|^2 + 4(n^2-4)A\widetilde{R}(\widetilde{\Gamma})}\right)^{(n-2)/2} - 1\Bigg),\end{aligned} \tag{57}$$

$$\lambda_{c0}(\Psi_0) = 6H_1^2 = 2\left(\frac{m_{\tilde{A}}}{m_{\text{pl}}}\right)^n \frac{m_{\tilde{A}}^2}{\alpha_s^{2(n+1)}} \cdot \frac{|\underline{\widetilde{P}}|^{(n+2)/2}n^{n/2}}{(n+2)^{(n+2)/2}\left(\widetilde{R}(\widetilde{\Gamma})\right)^{n/2}}. \tag{58}$$

The density ρ_r is of course an effective radiation density, because we adsorbe into ρ_r a factor with an effective gravitational constant.

In our simplified model we get

$$\begin{aligned}\rho_r = \Lambda\Bigg(&\frac{98g^2(\zeta,\mu) + (\mu^2+4)\sqrt{49g^2(\zeta,\mu) + 384h(\zeta,\mu)} + 32H(\zeta,\mu)}{g^8(\zeta,\mu)(\mu^2+4)^2\cdot 2^{11}\cdot 7^{14}} \\ &\times \left(7g(\zeta,\mu) + \sqrt{49g^2(\zeta,\mu) + 384h(\zeta,\mu)}\right)^6 \\ &\times \left(\ln\left(|\zeta| + \sqrt{\zeta^2+1}\right) + 2\zeta^2 + 1\right)^6 - 1\Bigg),\end{aligned} \tag{59}$$

$$
\begin{aligned}
\Lambda = {} & \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^{14} \cdot \frac{m_{\tilde{A}}^2}{\alpha_s^{36}} \cdot \frac{7^{14}}{2^{24}} \\
& \times \frac{g^8(\zeta,\mu)(\mu^2+4)^8}{(2\mu^3+7\mu^2+5\mu+20)^7\left(\ln\left(|\zeta|+\sqrt{\zeta^2+1}\right)+2\zeta^2+1\right)^8} .
\end{aligned}
\tag{60}
$$

In the formulas (59–60) we should put $m_{\tilde{A}} = m_{\rm EW}$ and $\alpha_s^2 = \alpha_{\rm em} = \frac{1}{137}$. Let us notice that Λ is our contemporary cosmological constant given by (4). It gives a scale for ρ_r.

The functions $g(\zeta,\mu)$, $h(\zeta,\mu)$, $H(\zeta,\mu)$ are given by Eqs (40), (41a), (41b), (42) from Ref. [1]. The numerical factor in front of Λ can be calculated and we get

$$
\Lambda = 1.04 \cdot 10^{-178}\, m_{\rm EW}^2 a(\zeta,\mu) \tag{61}
$$

where

$$
a(\zeta,\mu) = \frac{g^8(\zeta,\mu)(\mu^2+4)^8}{(2\mu^3+7\mu^2+5\mu+20)^7\left(\ln\left(|\zeta|+\sqrt{\zeta^2+1}\right)+2\zeta^2+1\right)^8} . \tag{62}
$$

The calculated radiation density evolves in time according to the theory developed in Ref. [4].

In order to find some influence of q_0 (quintessence) field on the value of the effective gravitational constant we consider a field equation for the scalar q_0 field in empty space. One gets

$$
\left(\frac{\partial^2 q_0}{\partial t^2} - \overrightarrow{\nabla}^2 q_0\right) + \widetilde{\varepsilon}\,\overline{\alpha}\exp(n\overline{\beta}q_0)\left(\exp(2\overline{\beta}q_0)-1\right) = 0 \tag{63}
$$

where

$$
\overline{\alpha} = \frac{\lambda_{c0} n(n+2) m_{\rm pl}^2}{16\pi \cdot \overline{M}} \tag{64}
$$

$$
\overline{\beta} = \frac{m_{\rm pl}}{2\sqrt{2\pi|\overline{M}|}} \tag{65}
$$

$$
\widetilde{\varepsilon} = \operatorname{sgn}\overline{M},\ \ \widetilde{\varepsilon}^2 = 1. \tag{66}
$$

Let us consider a static, spherically symmetric case. In the spherical coordinates one gets

$$
0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dq_0}{dr}\right) - \widetilde{\varepsilon}\,\overline{\alpha}\exp(n\overline{\beta}q_0)\left(\exp(2\overline{\beta}q_0)-1\right) \tag{67}
$$

where $q_0 = q_0(r)$ is a function of r only. In order to treat this equation it is easier to come back to the old variable $\varphi = \overline{\beta}q_0$ (see Eq. (28)). One gets

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) - \frac{1}{4}\widetilde{\varepsilon}\,\overline{\lambda}_{c0}\exp(n\varphi)\left(\exp(2\varphi)-1\right) = 0. \tag{68}
$$

We change the independent variable r into τ

$$\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2\frac{d\varphi}{d\tau}\right) - \widetilde{\varepsilon}\exp(n\varphi)\left(\exp(2\varphi)-1\right) = 0 \tag{69}$$

where

$$r = \frac{2}{\sqrt{\overline{\lambda}_{c0}}}\,\tau \tag{70}$$

and

$$\overline{\lambda}_{c0} = \frac{n(n+2)\lambda_{c0}}{8\pi\overline{M}}\,m_{\rm pl}^2\,. \tag{71}$$

We consider Eq. (69) in two regions:

1) for small fields φ,
2) for large fields φ.

In the first region we get

$$\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2\frac{d\varphi}{d\tau}\right) + \widetilde{\varepsilon}e^{n\varphi} = 0. \tag{72}$$

In the second region we get

$$\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2\frac{d\varphi}{d\tau}\right) - \widetilde{\varepsilon}e^{(n+2)\varphi} = 0. \tag{73}$$

Let us notice that both equations have similar nature and can be reduced to the equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \varepsilon\widetilde{\varepsilon}e^{y} = 0 \tag{74}$$

where in the first region $\varepsilon = 1$,

$$y = n\varphi, \tag{75}$$
$$x = \sqrt{n}\,\tau, \tag{76}$$

and in the second region $\varepsilon = -1$,

$$y = (n+2)\varphi, \tag{77}$$
$$x = \sqrt{n+2}\,\tau, \tag{78}$$

We can transform (74) into

$$x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + \varepsilon\widetilde{\varepsilon}xe^{y} = 0 \tag{79}$$

which is the celebrated Emden-Fowler equation known in the theory of gaseous spheres (see [5]). Let us notice that the first region (small fields) means large distances and the second region (large fields) means small distances.

In this way we should consider Eq. (79) in the region of small and large x. In the case of $\varepsilon\widetilde{\varepsilon} = 1$ the equation (74) has an exact solution

$$y = \ln\left(\frac{2}{x^2}\right). \tag{80}$$

Let us apply this to both regions (remembering that $\widetilde{\varepsilon}$ in both cases has a different sign).

One gets in the first region

$$q_0 = -\frac{2\sqrt{2\pi|\overline{M}|}}{m_{\rm pl}n}\ln\left(\frac{r}{\frac{2\sqrt{2}}{\sqrt{n\overline{\lambda}_{c0}}}}\right) \tag{81}$$

and

$$G_{\rm eff} = G_N\left(\frac{r}{\frac{2\sqrt{2}}{\sqrt{n\overline{\lambda}_{c0}}}}\right)^{(n+2)/n}. \tag{82}$$

In the second region

$$q_0 = -\frac{2\sqrt{2\pi|\overline{M}|}}{m_{\rm pl}(n+2)}\ln\left(\frac{r}{\frac{2\sqrt{2}}{\sqrt{(n+2)\overline{\lambda}_{c0}}}}\right) \tag{83}$$

and

$$G_{\rm eff} = G_N\left(\frac{r}{\frac{2\sqrt{2}}{\sqrt{(n+2)\overline{\lambda}_{c0}}}}\right). \tag{84}$$

In this way we get an interesting prediction for the behaviour of the strength of gravitational interactions. In this very special solution $G_{\rm eff}$ is going to zero if $r \to 0$ and to infinity if $r \to \infty$.

Let us come to Eq. (79) supposing $\varepsilon\widetilde{\varepsilon} = 1$. Thus we get

$$x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + xe^y = 0. \tag{85}$$

Using an exact solution (80) we write

$$y = y_1 + \widetilde{y} \tag{86}$$

and consider Eq. (85) for large x.

In this way we get an approximate solution (given by Chandrasekhar [5])

$$y=\ln\left(\frac{2}{\eta^2}\right)+\frac{A}{\sqrt{\eta}}\cos\left(\frac{\sqrt{7}}{2}\ln\eta\right)-2\ln\overline{\delta},\quad |A|\ll 1, \tag{87}$$

where $\eta=\frac{x}{\overline{\delta}}$, A and $\overline{\delta}$ are integration constants, $\overline{\delta}>0$. In this way we get in the first region

$$G_{\rm eff}=\overline{G}_{\rm eff}\cdot\exp\left(-\frac{\overline{A}}{\sqrt{\eta}}\cos\left(\frac{\sqrt{7}}{2}\ln\eta\right)\right) \tag{88a}$$

where $\overline{A}$ is a constant ($|\overline{A}|\ll 1$) and

$$\eta=\frac{\sqrt{n\overline{\lambda}_{c0}}\,\overline{\delta}}{2\sqrt{2}}\,r, \tag{88b}$$

$\overline{G}_{\rm eff}$ is given by the formula (82).

In the second region

$$G_{\rm eff}=\overline{\overline{G}}_{\rm eff}\cdot\exp\left(-\frac{\overline{\overline{A}}}{\sqrt{\eta}}\cos\left(\frac{\sqrt{7}}{2}\ln\eta\right)\right) \tag{88c}$$

where $\overline{\overline{A}}$ is a constant ($|\overline{\overline{A}}|\ll 1$), $\overline{\overline{G}}_{\rm eff}$ is given by the formula (84) and

$$\eta=\frac{\sqrt{(n+2)\overline{\lambda}_{c0}}\,\overline{\delta}}{2\sqrt{2}}\,r. \tag{88d}$$

In this way we have very interesting non-Newtonian behaviour of $G_{\rm eff}$ for large distances. Let us notice that the length scale is completely arbitrary, because it is given by an integration constant $\overline{\delta}$.

Let us consider Eq. (63) in Cartesian coordinates supposing flat symmetry for a quintessence field $q_0=q_0(z,t)$ (nonstatic). One gets

$$\left(\frac{\partial^2 q_0}{\partial t^2}-\frac{\partial^2 q_0}{\partial z^2}\right)-\widetilde{\varepsilon}\overline{\alpha}\exp(n\overline{\beta}q_0)\big(\exp(2\overline{\beta}q_0)-1\big)=0. \tag{89}$$

Let us change dependent and independent variables to ξ,η,φ:

$$z=\frac{2}{\sqrt{\overline{\lambda}_{c0}}}\,\xi \tag{90}$$

$$t=\frac{2}{\sqrt{\overline{\lambda}_{c0}}}\,\eta \tag{91}$$

$$\varphi=\overline{\beta}q_0. \tag{92}$$

One gets

$$\left(\frac{\partial^2\varphi}{\partial\eta^2}-\frac{\partial^2\varphi}{\partial\xi^2}\right)-\widetilde{\varepsilon}e^{n\varphi}\left(e^{2\varphi}-1\right)=0. \tag{93}$$

Eq. (93) is an equation for flat scalar (quintessence) waves in our theory. Let us consider it for large and small field φ (as before).

In this way one gets the equation

$$\left(\frac{\partial^2 y}{\partial T^2}-\frac{\partial^2 y}{\partial x^2}\right)+\varepsilon\widetilde{\varepsilon}e^{y}=0, \tag{94}$$

where in the region of small field φ we have $\varepsilon=1$ and

$$y=n\varphi \tag{95}$$
$$x=\sqrt{n}\,\xi \tag{96}$$
$$T=\sqrt{n}\,\eta \tag{97}$$

and in the region of large field φ, $\varepsilon=-1$ and

$$y=(n+2)\varphi \tag{98}$$
$$x=\sqrt{n+2}\,\xi \tag{99}$$
$$T=\sqrt{n+2}\,\eta. \tag{100}$$

Eq. (94) is the famous Liouville equation which can be transformed via a Bäcklund transformation into a two-dimensional wave equation and afterwards solved exactly. The general solution depends on two arbitrary functions f and g of one variable, sufficiently regular. It is possible to consider several problems for this equation: Cauchy initial problem, Darboux problem and Goursat problem.

The general solution of (94) looks like ($\varepsilon\widetilde{\varepsilon}=-1$)

$$y(T,x)=\ln\left[\frac{2g'(x-T)f'(x+T)}{\left(g(x-T)+f(x+T)\right)^2}\right] \tag{101}$$

where g' and f' are derivatives of g and f.

Thus one gets in the first region (small field)

$$q_0(t,z)=\frac{2\sqrt{2\pi\overline{M}}}{m_{\rm pl}n}\cdot\ln\left[\frac{2g'\left(\frac{z-t}{a}\right)f'\left(\frac{z+t}{a}\right)}{\left(g\left(\frac{z-t}{a}\right)+f\left(\frac{z+t}{a}\right)\right)^2}\right], \tag{102}$$

$$a=\frac{2\sqrt{2}}{\sqrt{n\overline{\lambda}_{c0}}}\,, \tag{103}$$

and

$$G_{\mathrm{eff}} = G_N \left(\frac{\left(g\left(\frac{z-t}{a}\right) + f\left(\frac{z+t}{a}\right)\right)^2}{2g'\left(\frac{z-t}{a}\right) f'\left(\frac{z+t}{a}\right)} \right)^{(n+2)/n} . \tag{104}$$

In the second region (large field)

$$q_0(t,z) = \frac{2\sqrt{2\pi\overline{M}}}{m_{\mathrm{pl}}(n+2)} \cdot \ln\left[\frac{2g'\left(\frac{z-t}{b}\right) f'\left(\frac{z+t}{b}\right)}{\left(g\left(\frac{z-t}{b}\right) + f\left(\frac{z+t}{b}\right)\right)^2} \right], \tag{105}$$

$$b = \frac{2\sqrt{2}}{\sqrt{(n+2)\overline{\lambda}_{c0}}}, \tag{106}$$

and

$$G_{\mathrm{eff}} = G_N \cdot \frac{\left(g\left(\frac{z-t}{b}\right) + f\left(\frac{z+t}{b}\right)\right)^2}{2g'\left(\frac{z-t}{b}\right) f'\left(\frac{z+t}{b}\right)}. \tag{107}$$

In this way we get a spatio-temporal pattern of changing the effective gravitational constant for small and large field regions. In both cases we have $\varepsilon\widetilde{\varepsilon} = -1$. However, in the small field region we have $\varepsilon = 1$ and because of this $\widetilde{\varepsilon} = -1$. In the case of large field region $\varepsilon = -1$ and $\widetilde{\varepsilon} = 1$. In order to be in line with our assumptions we should consider in the first case such functions f and g that the expression in (105) is small and for the second case vice versa.

Let us consider Eq. (63) in cylindrical coordinates supposing cylindrical symmetry for the field q_0, $q_0 = q_0(\rho)$. One gets

$$\frac{1}{\rho}\left(\rho \frac{dq_0}{d\rho}\right) - \varepsilon\overline{\alpha}\exp(n\overline{\beta}q_0)\left(\exp(2\overline{\beta}q_0) - 1\right) = 0. \tag{108}$$

This equation can be transformed into

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau \frac{d\varphi}{d\tau}\right) - \widetilde{\varepsilon}\exp(n\varphi)\left(e^{2\varphi} - 1\right) = 0, \tag{109}$$

$$\rho = \frac{2}{\sqrt{\overline{\lambda}_{c0}}}\,\tau. \tag{110}$$

As usual we consider Eq. (109) in two regions for small and large fields.

In the first region

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau \frac{d\varphi}{d\tau}\right) + \widetilde{\varepsilon}e^{n\varphi} = 0. \tag{111}$$

In the second region we get

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau \frac{d\varphi}{d\tau}\right) - \widetilde{\varepsilon}e^{(n+2)\varphi} = 0. \tag{112}$$

Both equations can be reduced to the equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\varepsilon\widetilde{\varepsilon}e^{y}=0, \tag{113}$$

where in the first region $\varepsilon = 1$ and

$$y = n\varphi \tag{114}$$
$$x = \sqrt{n}\,\tau \tag{115}$$

and in the second region $\varepsilon = -1$ and

$$y = (n+2)\varphi \tag{116}$$
$$x = \sqrt{n+2}\,\tau. \tag{117}$$

We can transform (113) into

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\varepsilon\widetilde{\varepsilon}xe^{y}=0 \tag{118}$$

which is the equation considered in [6] for $\varepsilon\widetilde{\varepsilon}=1$.

Following H. Lemke we write down a solution to Eq. (118) in a compact form in three cases (concerning an integration constant introduced by H. Lemke). We adopt his solutions to our problem.

1) $C = \kappa^2 > 0$, κ—arbitrary positive number:

$$y(x) = -2\ln\left(\frac{x}{\overline{a}}\left(\left(\frac{\overline{a}}{x}\right)^{\kappa}+\left(\frac{x}{\overline{a}}\right)^{\kappa}\right)\right)+\ln\left(4\kappa^2\overline{a}^2\right) \tag{119}$$

where $\overline{a} > 0$ is an arbitrary constant.

2) $C = -\omega^2 < 0$, ω is an arbitrary positive number:

$$y(x) = -2\ln\left(2x\sin(\omega\ln x+\delta)\right)+\ln(4\omega^2) \tag{120}$$

where δ is an arbitrary constant.

3) $C = 0$:

$$y(x) = -2\ln\left(\frac{x}{\overline{a}}\ln\left(\frac{x}{\overline{a}}\right)\right)-2\ln\overline{a} \tag{121}$$

where $\overline{a} > 0$ is an arbitrary constant.

Using these solutions we write down a spatial dependence of $G_{\rm eff}$ in the case of small and large fields. In the case of small fields one gets

1) $$G_{\rm eff} = G_N\left(4\kappa^2\overline{a}^2\right)^{(n+2)/n}\frac{\left(1+r^{2n}\right)^{2(n+2)/n}}{r^{2(\kappa-1)(n+2)/n}} \tag{122}$$

where

$$r = \frac{1}{2\overline{a}}\sqrt{n\overline{\lambda}_{c0}}\,\rho. \tag{123}$$

2)
$$G_{\text{eff}} = \frac{G_N(\overline{\delta})^{2(n+2)/n}}{(\omega^2)^{(n+2)/n}}\, r^{2(n+2)/n}\big(\sin(\omega\ln r)\big)^{2(n+2)/n} \tag{124}$$

where

$$r = \frac{1}{2\overline{\delta}}\sqrt{n\overline{\lambda}_{c0}}\,\rho \tag{125}$$

$$\omega\ln\overline{\delta} = -\delta \tag{126}$$

3)
$$G_{\text{eff}} = G_N\overline{a}^{2(n+2)/n} r^{2(n+2)/n}(\ln r)^{2(n+2)/n} \tag{127}$$

and r is given by Eq. (123).

In the case of large field one gets:

1)
$$G_{\text{eff}} = G_N(4\kappa^2\overline{a}^2)\frac{(1+r^{2\kappa})^2}{r^{2(\kappa-1)}} \tag{128}$$

where

$$r = \frac{1}{2\overline{a}}\sqrt{(n+2)\overline{\lambda}_{c0}}\,\rho. \tag{129}$$

2)
$$G_{\text{eff}} = G_N\left(\frac{\overline{\delta}}{\omega}\right)^2 r^2\big(\sin(\omega\ln r)\big)^2 \tag{130}$$

where

$$r = \frac{1}{2\overline{\delta}}\sqrt{(n+2)\overline{\lambda}_{c0}}\,\rho, \quad \ln\overline{\delta} = -\frac{\delta}{\omega}. \tag{131}$$

3)
$$G_{\text{eff}} = G_N\overline{a}^2 r^2(\ln r)^2 \tag{132}$$

and r is given by Eq. (129).

Let us notice that in that spatial dependence for large and small field we have κ and $\overline{a}$ ($\overline{\delta}$, ω) as integration constants. In this way integration constants induce a power law of this dependence and also a scale. For sufficiently big n ($n > 14$) there is no significant difference between both cases. It means the quintessence field behaves everywhere as for large field case (in these solutions of course). It is evident that the spatial dependence (in cylindrical symmetry case) of G_{eff} goes to some kind of the fifth force. However, we have to do not with a universal law of Nature but rather with some kind of initial conditions. Thus the real dependence of G_{eff} on spatial coordinates can

be obtained after averaging G_{eff} with respect to initial conditions. Let us describe it using Eq. (128).

Let us write G_{eff} in a form where κ and $\overline{a}$ are explicitly visible:

$$G_{\text{eff}} = G_N \left(4\kappa^2\overline{a}^2\right) \cdot \frac{\left(1 + \left(\frac{x}{\overline{a}}\right)^{2\kappa}\right)^2}{\left(\frac{x}{\overline{a}}\right)^{2(\kappa-1)}}, \tag{133}$$

$$x = \frac{1}{2}\sqrt{(n+2)\overline{\lambda}_{c0}}\,\rho. \tag{134}$$

The constants $\overline{a}$ and κ are chosen randomly for they are dependent on initial conditions. Let μ be a measure defined on $(0,+\infty)^2$, positive and normalized to 1, i.e.

$$\int_0^{+\infty}\int_0^{+\infty} d\mu(\overline{a},\kappa) = 1. \tag{135}$$

This measure gives an account how frequently we have to do with some initial conditions (i.e. with integration constants $\overline{a}$, κ). In this way an experimental dependence of G_{eff} on x should be obtained from

$$E(G_{\text{eff}}) = 4G_N \int_0^{+\infty}\int_0^{+\infty} \frac{\kappa^2\overline{a}^2\left(1 + \left(\frac{x}{\overline{a}}\right)^{2\kappa}\right)^2}{\left(\frac{x}{\overline{a}}\right)^{2(\kappa-1)}}\, d\mu. \tag{136}$$

It means it is an expectation value of G_{eff} with respect to the measure μ. μ need not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. The formula (136) could give an account on some serious problems with comparisons of several measurements of G (a Newton constant). Maybe we have to do with different initial conditions for quintessence field. These initial conditions appear with different probabilities according to the measure $d\mu(\overline{a},\kappa)$ going to an expectation value $E(G_{\text{eff}})$. The second central moment of $d\mu(\overline{a},\kappa)$ (if it exists) can express a deviation from the law described by $E(G_{\text{eff}})$.

In the simplest case we can suppose a continuous Gaussian distribution for κ only:

$$d\mu(\kappa) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(\kappa-\kappa_0)^2/(2\sigma^2)}\, d\kappa.$$

In this case we have

$$\overline{G}_{\text{eff}}(x) = E(G_{\text{eff}}(x)) = \frac{G_N\overline{a}^2}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} d\kappa\, e^{-(\kappa-\kappa_0)^2/(2\sigma^2)}\kappa^2\, \frac{\left(1 + \left(\frac{x}{\overline{a}}\right)^{2\kappa}\right)^2}{\left(\frac{x}{\overline{a}}\right)^{2(\kappa-1)}}\,. \tag{136a}$$

One gets after some algebra

$$
\begin{aligned}
\overline{G}_{\text{eff}}(x) &= G_N \kappa_0 \exp\left(\overline{a} - \frac{\kappa_0^2}{2\sigma^2}\right)\left(\frac{x}{\overline{a}}\right)^2 \\
&\times \Bigg[4\sigma^4 \exp \frac{\left(2\ln\left(\frac{x}{\overline{a}}\right)\sigma^2 + \kappa_0^2\right)^2}{2\sigma^2} \ln\left(\frac{x}{\overline{a}}\right) \\
&\quad + (2\sigma^2 + \kappa_0^4)\exp\frac{\left(2\ln\left(\frac{x}{\overline{a}}\right)\sigma^2 + \kappa_0^2\right)^2}{2\sigma^2} \\
&\quad - 4\sigma^4 \exp\frac{(\sigma+\kappa_0)^2}{2}\ln^2\left(\frac{x}{\overline{a}}\right) + 2(\sigma^2+\kappa_0^2)\exp\frac{(\sigma+\kappa_0)^2}{2}\Bigg].
\end{aligned}
\tag{136b}
$$

In the case of the solution (107) parametrized by two functions of one variable we can consider them as random variables parametrized by z and t. Moreover in this case we should consider a measure μ on an infinite-dimensional space of functions f and g, supposing that solution (107) is generalized to this space. Afterwards we can use as μ the Wiener or Gaussian measure in L^2 space. If we use for Gaussian distribution the normalized distribution $N(0,1)$, the formula (136b) simplifies to

$$
\begin{aligned}
\overline{G}_{\text{eff}}(x) &= e^{\bar{a}}\sqrt{e}\,G_N \\
&\times \left[4\left(\frac{x}{\overline{a}}\right)^{2(1+\ln(x/\bar{a}))}\ln\left(\frac{x}{\overline{a}}\right) + \left(\frac{x}{\overline{a}}\right)^{2(1+\ln(x/\bar{a}))} - 4\ln^2\left(\frac{x}{\overline{a}}\right) + 2\right].
\end{aligned}
\tag{136c}
$$

Let us consider three our cases (119), (120) and (121) for small and large fields cases.

The small field case is such that

$$e^{\varphi} < 1, \qquad \text{i.e. } \varphi < 0. \tag{137}$$

The large field case is if

$$e^{\varphi} > 1, \qquad \text{i.e. } \varphi > 0. \tag{138}$$

For (119) we have for the small case

$$f(r) > 1,$$

where

$$f(r) = r^{1-\kappa} + r^{\kappa+1}, \qquad r = \left(\frac{x}{\overline{a}}\right). \tag{139}$$

Let us consider two cases

$$0 < \kappa < 1 \qquad \text{and} \qquad \kappa > 1.$$

In the first case $f(r) \nearrow$ in $[0, +\infty)$ and we have simply $r > r_0$ where r_0 satisfies the equation

$$r_0^{2\kappa+1} + r_0 - 1 = 0. \tag{140}$$

In the second case $\kappa > 1$,

$$\lim_{r \to 0} f(r) = +\infty \qquad \text{and} \qquad \lim_{r \to \infty} f(r) = +\infty.$$

The function $f(r)$ has a minimum at

$$r_1 = \left(\frac{\kappa - 1}{\kappa + 1} \right)^{1/\kappa}. \tag{141}$$

Let us calculate $f(r_1)$.

$$f(r_1) = \left(\frac{\kappa + 1}{\kappa - 1} \right)^{\kappa - 1/\kappa} + \left(\frac{\kappa - 1}{\kappa + 1} \right)^{\kappa + 1/\kappa}. \tag{142}$$

It is easy to see that if $\kappa > 1$, then $\kappa - 1/\kappa > 0$. This means that

$$f(r_1) > 1 \tag{143}$$

and therefore $f(r) > 1$ for all $r > 0$. Thus simultaneously we get a solution for large field only if $\kappa < 1$, i.e.

$$r < r_0. \tag{144}$$

If $\kappa = 1$, we always have $f(r) \geq 1$ (i.e. only a small field).

Let us consider (120). In this case the small field condition reads

$$h(r) = r \sin(\omega \ln r) > 1, \quad \text{where } r = \frac{x}{\bar{\delta}}. \tag{145}$$

First of all we need $h(r) > 0$. Let us observe that $|h(r)| \leq r$ for every $r > 0$. Next we see that the roots of $h(r)$ are the numbers

$$r_{0,k} = e^{k\pi/\omega}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{146a}$$

and

$$h(r) > 0 \quad \text{if} \quad r_{0,2k} < r < r_{0,2k+1}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{146b}$$

The maximum of the function $h(r)$ in the interval $(r_{0,2k}, r_{0,2k+1})$ is greater than

$$h(r_{0,2k+1/2}) = e^{2k\pi/\omega} \cdot e^{\pi/(2\omega)}, \tag{147}$$

but smaller than $r_{0,2k+1}$. Thus the maxima are smaller than 1 for negative k and greater than 1 if $k = 0, 1, 2, \ldots$. It means that in each interval $(r_{0,2k}, r_{0,2k+1})$, $k = 0, 1, 2, \ldots$, there exist two numbers $r_{3,2k}$ and $r_{2,2k}$ such that $r_{3,2k} < r_{2,2k}$,

$$h(r_{3,2k}) = h(r_{2,2k}) = 1 \tag{148}$$

and

$$h(r) > 1 \quad \text{if} \quad r_{3,2k} < r < r_{2,2k}\,. \tag{149}$$

The condition for large fields, $0 < h(r) < 1$, is satisfied if

$$r_{0,2k} < r < r_{0,2k+1}, \quad k = -1, -2, \ldots \tag{150}$$

or

$$r_{0,2k} < r < r_{3,2k}, \quad \text{or} \quad r_{2,2k} < r < r_{0,2k+1}, \quad k = 0, 1, 2, \ldots \tag{151}$$

In the third case, i.e. Eq. (121), one has for the small field case

$$r \ln r > 1 \tag{152}$$

where $r = \frac{x}{a}$. Let $r_4 \ln r_4 = 1$, $r_4 > 1$. We have $r > r_4 = 1.7632\ldots$. In the large field case

$$0 < r < r_4 = 1.7632\ldots\,. \tag{153}$$

Thus we have in general large fields on large distances.

In this way we have solutions for large and small distances. One can try to connect them to get a solution for all distances. However in this case it is necessary to be very careful, for our solutions depend on some integration constants which can be different for both asymptotic regions.

Let us consider Eq. (63) in two special cases:

I $q_0 = q(z)$ — static and depending only on z;

II $q_0 = q_0(t)$ — non-static and spatially constant.

Let us consider also these cases for small and large fields φ. In all of these cases we come to the following equation

$$\frac{d^2y}{dx^2} + \varepsilon_1 \varepsilon \widetilde{\varepsilon} e^y = 0 \tag{154}$$

where $\varepsilon_1 = 1$ for case I and $\varepsilon_1 = -1$ for case II.

Eq. (154) can easily be reduced to the integral

$$x - x_0 = \frac{\varepsilon_3}{\sqrt{2}} \int \frac{dy}{\sqrt{\varepsilon_2 \omega^2 - \eta e^y}} \tag{155}$$

where $\eta = \varepsilon_1 \varepsilon \widetilde{\varepsilon}$, $\eta^2 = 1$, $C = 2\varepsilon_2 \omega^2$, $\omega \geq 0$, $\varepsilon_2^2 = 1$ is an integration constant, $\varepsilon_3^2 = 1$ and x_0 also is an integration constant.

After some calculation we get the following solutions:

A. $$y(x) = 2\left[\ln\omega - \ln\left|\sinh\left(\frac{\sqrt{2}\,(x-x_0)\omega}{2}\right)\right|\right], \quad \eta=-1,\ \varepsilon_2=1 \tag{156}$$

B. $$y(x) = 2\left[\ln\omega - \ln\left|\sinh\left(\frac{\sqrt{2}\,(x-x_0)\omega}{2}\right)\right|\right], \quad \eta=1,\ \varepsilon_2=1 \tag{157}$$

where

$$\frac{\sqrt{2}\,(x-x_0)\omega}{2} > \ln(1+\sqrt{2}) \tag{158}$$

or

$$\frac{\sqrt{2}\,(x-x_0)\omega}{2} < \ln(\sqrt{2}-1). \tag{159}$$

C. $$y(x) = 2\left[\ln\omega - \ln\left|\cos\left(\frac{\sqrt{2}\,(x-x_0)\omega}{2}\right)\right|\right], \quad \eta=-1,\ \varepsilon_2=-1. \tag{160}$$

Let us apply these solutions to our problems. First of all let us consider a static configuration with z dependence only. In this case $\varepsilon_1 = 1$ and $\eta = \varepsilon\widetilde{\varepsilon}$. For the small field case one gets, $\varepsilon = 1$, $\eta = \widetilde{\varepsilon}$,

$$G_{\rm eff} = \frac{G_N}{\omega^{2(n+2)/n}}\,|\sinh p|^{(n+2)/n} \tag{161}$$

where

$$p = \frac{\omega(z-z_0)}{4}\sqrt{n\overline{\lambda}_{c0}}\,. \tag{162}$$

For the large field case we get, $\varepsilon = -1$, $\eta = -\widetilde{\varepsilon}$,

$$G_{\rm eff} = \frac{G_N}{\omega^2}\,|\cos p| \tag{163}$$

where

$$p = \frac{\omega(z-z_0)}{4}\sqrt{(n+2)\overline{\lambda}_{c0}}\,. \tag{164}$$

In this case $\widetilde{\varepsilon} = 1$ for $\eta = -1$.

In a nonstatic configuration $\varepsilon_1 = -1$ and $\eta = -\varepsilon\widetilde{\varepsilon}$. For the small field case ($\varepsilon = 1$), $\eta = -\widetilde{\varepsilon}$,

$$G_{\rm eff} = \frac{G_N}{\omega^{2(n+2)/n}}\,|\sinh q|^{(n+2)/n} \tag{165}$$

where

$$q = \frac{\varepsilon_3\omega(t-t_0)}{4}\sqrt{n\overline{\lambda}_{c0}} \tag{166}$$

and analogically for the large field case ($\varepsilon = -1$), $\eta = \widetilde{\varepsilon}$,

$$G_{\text{eff}} = \frac{G_N}{\omega^2} \left|\cos q\right|, \tag{167}$$

$$q = \frac{\omega(t - t_0)}{4} \sqrt{(n+2)\overline{\lambda}_{c0}}\,. \tag{168}$$

In this case $\widetilde{\varepsilon} = -1$ for $\eta = -1$.

Let us notice that in a static configuration for small field we have two possibilities for $\eta = -1$ (no condition on p) and $\eta = 1$ (conditions (158–159)). Thus without conditions we have $\widetilde{\varepsilon} = -1$ and with conditions $\widetilde{\varepsilon} = 1$. In a non-static configuration for small field we have vice versa $\widetilde{\varepsilon} = 1$ without conditions and $\widetilde{\varepsilon} = -1$ with conditions (158–159).

Let us come back to the Eq. (89) and consider it in a travelling wave scheme. In this way we have

$$q_0(z,t) = \widetilde{q}(z - vt) \tag{169}$$

where v is a velocity of the travelling wave (a soliton), $|v| < 1$. Let us consider this equation in both regimes (for small and large fields). In this way we come to the expression

$$(1 - v^2)\,\frac{d^2\chi}{d\xi^2} - \varepsilon\widetilde{\varepsilon}e^{\chi} = 0 \tag{170}$$

where χ is a shape function of a soliton. Changing an independent variable from ξ to λ one gets

$$\frac{d^2\chi}{d\lambda^2} - \varepsilon_1\varepsilon\widetilde{\varepsilon}e^{\chi} = 0 \tag{171}$$

where $\varepsilon_1 = -1$, i.e. we get Eq. (154) with $\eta = -\varepsilon\widetilde{\varepsilon}$,

$$\lambda = \frac{\xi}{\sqrt{1 - v^2}}, \quad \xi = \sqrt{1 - v^2}\,\lambda. \tag{172}$$

In this way we adopt our solutions A, B, C in both regimes: small and large field (changing χ into λ). For small field we get ($\varepsilon = 1$, $\eta = -\widetilde{\varepsilon}$)

$$q_0(z,t) = \frac{2}{\overline{\beta}n}\left[\ln\omega - \ln\left|\sinh p\right|\right] \tag{173}$$

where

$$p = \frac{\omega\sqrt{n\overline{\lambda}_{c0}}}{4\sqrt{1 - v^2}}\,(z - vt). \tag{174}$$

For large field we get ($\varepsilon = -1$, $\eta = \widetilde{\varepsilon}$)

$$q_0(z,t) = \frac{2}{\overline{\beta}(n+2)}\left[\ln\omega - \ln\left|\cos p\right|\right] \tag{175}$$

where

$$p = \frac{\omega\sqrt{(n+2)\overline{\lambda}_{c0}}}{4\sqrt{1-v^2}}\,(z - vt). \tag{176}$$

For (173) we have $\eta = -\widetilde{\varepsilon}$ and because of this $\widetilde{\varepsilon} = 1$ without any conditions and if $\widetilde{\varepsilon} = -1$ we have conditions (158–159). In the case of the formula (174) $\eta = -1$ and $\widetilde{\varepsilon} = -1$.

We can write down formulas for G_{eff} in the soliton case

$$G_{\mathrm{eff}} = \frac{G_N}{\omega^{2(n+2)/n}}\,|\sinh p|^{(n+2)/n} \tag{177}$$

and p is given by the formula (174) (with or without conditions (158–159)). This is of course a small field case.

In the large field case

$$G_{\mathrm{eff}} = \frac{G_N}{\omega^2}\,|\cos p|\,, \tag{178}$$

and p is given by the formula (176). In this case $\widetilde{\varepsilon} = -1$. (Let us notice that this is a case of SO(3) group in our theory.)

Let us notice that conditions (158–159) can be considered as conditions for small field in z or t domains. Let us notice that in our solutions concerning a behaviour of an effective gravitational constant we get completely arbitrary length or time scale (given by integration constants). In this way a spatial or time dependence of G_{eff} can be (except the solution (80) and simultaneously the approximate solution in the case of spherical symmetry) such that G_{eff} can be really constant on distances (or times) accessible in experiments. Only a statistical approach mentioned here can give a light on this dependence to compare it with an experiment.

In order to connect our results to the ordinary gravitational physics we consider again Eq. (67) in small field regime for initial conditions $\varphi(0) = 0$ and $\frac{d\varphi}{dx}(0) = 0$. The first condition means that we want to have $G_{\mathrm{eff}}(0) = G_N$ and the second that the quintessence field does not grow quickly. The problem cannot be solved analytically. Moreover R. Emden in the first point of Ref. [5] did it for us. We quote here his results adopted to our notation ($\varepsilon = +1$ and $\widetilde{\varepsilon} = -1$).

x	$-y$	e^y	G_{eff}/G_N
0.00	0.00000	1.00000	1.00000
0.25	0.01037	0.98969	0.98823
0.50	0.04113	0.95971	0.95409
0.75	0.09113	0.91290	0.90109
1.00	0.15903	0.85296	0.83380
1.25	0.24225	0.78486	0.75816
1.50	0.33847	0.71285	0.67920
1.75	0.44488	0.64090	0.60143

x	$-y$	e^y	G_{eff}/G_N
2.00	0.55967	0.57140	0.52749
2.50	0.80584	0.44671	0.39813
3.00	1.06226	0.34537	0.29670
3.50	1.31937	0.26730	0.22138
4.00	1.57071	0.20790	0.16611
4.50	1.81246	0.16325	0.12601
5.00	2.04264	0.12968	0.09686
6.00	2.46598	0.08493	0.05971
7.00	2.84160	0.05833	0.03887
8.00	3.17489	0.04180	0.02656
9.00	3.47128	0.03108	0.01893
10.00	3.73646	0.02384	0.01398
100	8.59506	0.000175	$5.0854 \cdot 10^{-5}$
1000	13.09847	0.000002	$3.0683 \cdot 10^{-7}$

where

$$x = \frac{1}{2}\sqrt{n\overline{\lambda}_{c0}}\, r \cong \left(\frac{r}{10\,\text{Mpc}}\right), \tag{179}$$

$$\frac{G_{\text{eff}}}{G_N} = (e^y)^{(n+2)/n} = (e^y)^{8/7}. \tag{180}$$

We take $n = 14$.

It is easy to see that for large n

$$\frac{G_{\text{eff}}}{G_N} = e^y. \tag{181}$$

It is easy to see that on a distance of 1 Mpc G_{eff} does not differ from G_N. Even on a distance of 10 Mpc it is about 10% smaller. Thus in the Solar System Newtonian gravitational physics does not change. Even on the level of a galaxy this change is minimal and cannot be observed. Moreover, there is an important conclusion: on distances about 200 Mpc the strength of gravitational interactions is about 10^{-5} times this on short distances measured in the Solar System (and for 10^3 Mpc of 10^{-7}). It is hard to tell how it influences a mass of a cluster of galaxies if we realize that from any observational data only a product GM has been obtained (not M).

From the other side on distances of 100 Mpc the strength of gravitational interactions is very weak (not only because of the distance). Thus if we consider clusters of galaxies as substrat particles in cosmology then they do not interact.

Let us consider Eq. (63) in Cartesian coordinates for two-dimensional static case (i.e. $\frac{\partial}{\partial z} = 0$, $\frac{\partial}{\partial t} = 0$). One gets

$$\left(\frac{\partial^2 q_0}{\partial x^2} + \frac{\partial^2 q_0}{\partial y^2}\right) - \widetilde{\varepsilon}\overline{\alpha}\exp(n\overline{\beta}q_0)\bigl(\exp(2\overline{\beta}q_0) - 1\bigr) = 0 \tag{182}$$

(where $\overline{\alpha}$, $\overline{\beta}$ are given by formulas (64) and (65)).

As usual, we come to the formula

$$\left(\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}\right)-\widetilde{\varepsilon}e^{n\varphi}(e^{2\varphi}-1)=0 \tag{183}$$

where

$$\left.\begin{matrix} x \\ y \end{matrix}\right\}=\frac{2}{\sqrt{\overline{\lambda}_{c0}}}\,x_i, \qquad i=1,2. \tag{183a}$$

We consider Eq. (183) for small and large fields and we get

$$\left(\frac{\partial^2\chi}{\partial z_1^2}+\frac{\partial^2\chi}{\partial z_2^2}\right)-\varepsilon\widetilde{\varepsilon}e^{\chi}=0 \tag{184}$$

where as usual for a small field $\varepsilon=1$ and

$$\chi=n\varphi, \tag{185}$$

$$z_i=\sqrt{n}\,x_i \tag{186}$$

and for a large field $\varepsilon=-1$ and

$$\chi=(n+2)\varphi, \tag{187}$$

$$z_i=\sqrt{n+2}\,x_i\,. \tag{188}$$

Thus we come to the equation known as Liouville equation

$$\Delta\chi=e^{\chi} \tag{184a}$$

if $\varepsilon\widetilde{\varepsilon}=1$.

This equation can be explicitly solved. First of all we change independent variables into

$$Z=\frac{1}{\sqrt{2}}(z_1+iz_2) \tag{189}$$

$$\chi(Z)=-\ln\left(\tfrac{1}{2}(1-|g|^2)\right)+\tfrac{1}{2}\ln\left|\frac{dg}{dZ}\right| \tag{190}$$

where g is an arbitrary analytic function on a complex plane Z.

In this way we get for the small field case

$$G_{\rm eff}=G_N\left(\frac{1-|g(Z)|^2}{2}\right)^{(n+2)/n}\left(\left|\frac{dg}{dZ}(Z)\right|\right)^{-(n+2)/(2n)} \tag{191}$$

where

$$Z = \sqrt{\frac{\overline{\lambda}_{c0}}{2n}} \cdot (x+iy). \tag{192}$$

In the large field case

$$G_{\mathrm{eff}} = G_N \left(\frac{1-|g(Z)|^2}{2}\right)\left(\left|\frac{dg}{dZ}(Z)\right|\right)^{-1/2} \tag{193}$$

where

$$Z = \sqrt{\frac{\overline{\lambda}_{c0}}{2(n+2)}} \cdot (x+iy). \tag{194}$$

Eqs (191) and (193) can have very interesting behaviour for $\frac{dg}{dZ}$ could have some singularities. The physical interpretation of these singularities can be very interesting.

Similarly as for Eqs (104) and (107) we can consider an expectation value of G_{eff} with respect to some kind of normalized measure for a space of analytic functions on the complex plane (respectively chosen).

It seems that an assumption all the energy of a quintessence is stored as quintessence particles is unrealistic. Let us suppose that only a fraction of this energy is stored as particles. Let this fraction be η,

$$0 < \eta < 1. \tag{195}$$

η can be a function of time and even of a space-point (locally).

Let us suppose that a gas of quintessence particles is a perfect gas governed by the Clapeyron equation. Thus we have

$$\frac{p_1}{\rho_1} = \frac{K_B T}{m_0} = \frac{T}{T_0}, \quad T_0 = \frac{m_0}{K_B} \simeq 0.11\,^\circ\mathrm{K} \tag{196}$$

(if we take $m_0 \simeq 10^{-5}\,\mathrm{eV}$). T is the temperature of a gas.

One gets

$$\rho_1 = \eta\rho_Q = \eta\rho \tag{197}$$

$$p = p_Q + p_1 = p_Q + \eta\,\frac{T}{T_0}\,\rho = -\rho(1-\eta) + \rho\,\frac{T}{T_0}\,\eta. \tag{198}$$

Eventually one gets

$$p = \rho\left(\eta\left(1+\frac{T}{T_0}\right)-1\right) \tag{199}$$

$$p = w\rho \tag{200}$$

$$w = \eta\left(1+\frac{T}{T_0}\right)-1. \tag{201}$$

Now we can calculate an isothermic speed of sound in a quintessence.

$$C_1^2 = \frac{p}{\rho} = w = \eta\left(1 + \frac{T}{T_0}\right) - 1. \tag{202}$$

For

$$0 < C_1^2 < 1 \tag{203}$$

we get

$$\frac{T_0}{T + T_0} < \eta < \min\left[\frac{2T_0}{T_0 + T}, 1\right]. \tag{204}$$

Let us remind to the reader that an isothermic sound is appropriate for low frequency of acoustic waves (we have to do with this sound in astrophysics). In general we have to do with so called adiabatic sound. In order to calculate a speed of an adiabatic sound in a quintessence we should find an analogue for a Poisson adiabate for our equation of state. One gets supposing that an internal energy of a quintessence is an energy of one-atomic gas of quintessence particles

$$dU + p\,dV = 0 \tag{205}$$

or

$$\frac{3}{2}\,dT = \frac{d\rho}{\rho}\left(\eta - 1 + \frac{T}{T_0}\,\eta\right) \tag{206}$$

and finally

$$\frac{p}{\rho^\kappa} = \text{const.} \tag{207}$$

$$\kappa = \frac{2\eta + 3}{3} \tag{208}$$

and a speed of an adiabatic sound simply reads

$$C_2 = \sqrt{\frac{\kappa p}{\rho}} = \sqrt{\kappa\left(\eta\left(1 + \frac{T}{T_0}\right) - 1\right)}. \tag{209}$$

It is interesting to ask what kind of a polythrope κ represents. Let us remind to the reader that in general

$$\kappa = \frac{c_p - c}{c_v - c} \tag{210}$$

where c is a specific heat of a polythrope in mind.

One gets in our case

$$c = \frac{3(\eta - 1)}{2\eta}\,\overline{R} \tag{211}$$

where $\overline{R}$ is a universal gas constant.

Thus we found both speeds of sound in a quintessence for low frequency and for high frequency of sound. The measurement of both speeds can help us to find η and T. One gets

$$\eta = \frac{3}{2}\left(\left(\frac{C_2}{C_1}\right)^2 - 1\right) \tag{212}$$

and

$$T = \frac{T_0}{3(C_2^2 - C_1^2)}\left(C_1^2(2C_1^2 + 5) - 3C_2^2\right). \tag{213}$$

Let us notice that a gas of quintessence particles is quite cold. Moreover these particles are highly relativistic. For the temperature $T \simeq T_0 = 0.11\,^\circ\mathrm{K}$ one sees that a speed of a quintessence particle is about 0.91 of the speed of light. Moreover we can consider lower temperatures. It is interesting to notice that the speed of sound is of the same order.

Thus if we want to have both speeds smaller than a speed of light,

$$0 < C_i^2 < 1, \quad i = 1, 2, \tag{214}$$

we get

$$\eta > \frac{1 + \sqrt{13 + 12t}}{1 + t} = f(t) \tag{215}$$

where

$$t = \frac{T}{T_0}. \tag{216}$$

The condition

$$t = \frac{T}{T_0} > 13 \tag{217}$$

guarantees that $f(t) < 1$. Moreover for $t = 14$ one gets

$$\eta \geq 0.96357. \tag{218}$$

This seems quite interesting, however maybe too much. Let us calculate a mean scattering length of quintessence particles for such a big $\eta = 0.96357$

$$l_{\text{scattering}} = \frac{1}{\eta\sigma n} = 1.0378 \cdot l. \tag{219}$$

We have still to do with a very dense gas of quintessence particles.

It seems that it would be reasonable to repeat these calculations using a different equation of state for quintessence particles gas. In particular we can consider a gas

of quintessence particles as a massless boson gas (with spin zero). In this case the equation of state looks:

$$p = p_Q + p_1 = -(1-\eta)\rho + \frac{4\eta\sigma\rho}{3} T^4 \tag{220}$$

and an adiabate equation

$$dU = \overline{c}_v \, dT + p \, dV = 0 \tag{221}$$

where

$$\overline{c}_v = 16\sigma T^3, \tag{222}$$

σ is the Stefan-Boltzmann constant.

One easily integrates

$$\frac{p}{\rho^{\overline{\kappa}}} = \text{const.} \tag{223}$$

where

$$\overline{\kappa} = \frac{3+\eta}{3} \, . \tag{224}$$

We have as before two kinds of sound: an isothermic sound

$$\overline{C}_1 = \sqrt{\frac{p}{\rho}} = \sqrt{\eta - 1 + \frac{4\eta\sigma}{3} T^4} \tag{225}$$

and an adiabatic sound

$$\overline{C}_2 = \sqrt{\overline{\kappa}\left(\eta - 1 + \frac{4\eta\sigma}{3} T^4\right)} = \sqrt{\frac{\overline{\kappa} p}{\rho}} \, . \tag{226}$$

From $0 < \overline{C}_i^2 < 1$, $i = 1, 2$, one gets

$$\frac{1}{1 + \left(T/\overline{T}_0\right)^4} < \eta < \min\left[1, \frac{2}{1 + \left(T/\overline{T}_0\right)^4}\right] \tag{227}$$

and

$$\eta > \frac{-3t^4 - 1 + \sqrt{9t^8 + 18t^4 + 13}}{2(1+t^4)} = \overline{f}(t) \tag{228}$$

with the condition

$$t = \frac{T}{\overline{T}_0} > \frac{1}{\sqrt{2}} \simeq 0.707. \tag{229}$$

This condition guarantees that

$$\overline{f}(t) < 1 \tag{230}$$

where $T_0^4 = \frac{3}{4\sigma}$ or

$$\overline{T}_0 = \frac{1}{K_B} \sqrt[4]{\frac{360}{\pi}} \simeq \frac{3.27}{K_B}\,, \tag{231}$$

K_B is the Boltzmann constant.

From the measurements of $\overline{C}_1$ and $\overline{C}_2$ we can obtain as before η and T. One gets:

$$\eta = 3\left(\left(\frac{\overline{C}_2}{\overline{C}_1}\right)^2 - 1\right) \tag{232}$$

and

$$T = \frac{\overline{T}_0}{\sqrt[4]{3}} \left(\frac{4\overline{C}_1^2 + \overline{C}_1^4 - 3\overline{C}_2^2}{\overline{C}_2^2 - \overline{C}_1^2}\right)^{1/4}\,. \tag{233}$$

Recently many papers have appeared concerning the speed of sound in a quintessence (see Ref. [7]). In some of them the authors propose to measure this speed.

There are interesting propositions to include primordial gravitational waves to fluctuations of a quintessence and vice versa. We examine these problems below.

We consider a mass of a quintessence particle, a speed of sound in a quintessence and several solutions to quintessence equations coming to the interesting behaviour of an effective gravitational constant. Some of issues of our theory are considered in self-interacting Brans-Dike theory [8].

The fraction η can be connected with the part of an energy density of quintessence field Q in such a way that it is a fraction of fluctuation energy density around an equilibrium Q_0. In this way the q_0 field which can evolve due to an evolution of Q and due to fluctuations of primordial gravitational waves will be a source of a gas of thermalized particles. It seems that an approach with a boson equation of state is more appropriate to consider.

It is interesting to consider a fraction of a dark energy stored as boson particles as a dark matter in the Universe.

Let us consider a primordial spectrum of gravitational waves in our approach. This spectrum is flat for our Hubble constant is really constant

$$P_{\rm grav} = \frac{2}{M_{\rm pl}^2}\left(\frac{H_0}{2\pi}\right)^2 \tag{235}$$

(see Ref. [9]). Thus

$$n_{\rm gr} = 1. \tag{236}$$

Now we start to examine quintessence fluctuations caused by fluctuations of a metric. In order to do it we follow Ref. [9] to perturb a spatial part of a metric

$$ds^2 = R^2(\tau)\left[-d\tau^2 + \left(\delta_{ij} + 2\widetilde{E}_{ij}^T\right) dx^i\, dx^j\right], \tag{237}$$

τ is a conformal time.

If we expand Einstein equations for (237) up to linear terms, we get

$$\frac{d^2 E^T_{ij\vec{k}}}{d\tau^2} + 2RH_K \frac{dE^T_{ij\vec{k}}}{d\tau} + k^2 E^T_{ij\vec{k}} = 0, \quad K = 0, 1, \tag{238}$$

where $\widetilde{E}^T_{ij}$ is a spatial perturbation of the metric, $R(\tau)$ is a scale factor depending on a conformal time, $E^T_{ij\vec{k}}$ is a $\vec{k}$ Fourier component $\widetilde{E}^T_{ij}$

$$\begin{aligned} \widetilde{E}^T_{ij} &= \frac{1}{(2\pi)^{3/2}} \int E^T_{ij\vec{k}} e^{-i\vec{k}\vec{r}} \, d^3\vec{k} \\ \vec{k} &= (k_1, k_2, k_3), \quad |\vec{k}|^2 = k^2, \quad \vec{r} = (x, y, z). \end{aligned} \tag{239}$$

The important quantity is an amplitude of gravitational wave corresponding to E^T_{ij}, i.e.

$$h_{ij\vec{k}} = RE^T_{ij\vec{k}}. \tag{240}$$

H_K is a Hubble constant. We take $K = 1$. The relation between a conformal time τ and ordinary time t is given by

$$R(\tau) = -\frac{1}{H_1 \tau}, \quad -\infty < \tau < 0, \tag{241}$$

$$R(t) = R_1 e^{H_1 t}. \tag{242}$$

For $h_{ij\vec{k}}$ one gets

$$\frac{d^2 h_{ij\vec{k}}}{d\tau^2} - \frac{2}{\tau^2} \frac{dh_{ij\vec{k}}}{d\tau} + k^2 h_{ij\vec{k}} = 0. \tag{243}$$

This equation can be easily solved. For further investigations we need only

$$h_{\vec{k}} = h^i_{i\vec{k}} \tag{244}$$

$$h_{\vec{k}} = A_{\vec{k}} \left(\tau|k|\cos(\tau|k|) - \sin(\tau|k|)\right) + B_{\vec{k}} \left(\tau|k|\sin(\tau|k|) + \cos(\tau|k|)\right) \tag{245}$$

and especially

$$\frac{dh_{\vec{k}}}{d\tau} = A_{\vec{k}} \tau \sin(\tau|k|) + B_{\vec{k}} \tau \cos(\tau|k|) \tag{246}$$

where $A_{\vec{k}}$ and $B_{\vec{k}}$ are constants.

We need also a quintessence field time dependence (e.g. in a slow roll approximation). We use our solution from the last point of Ref. [2] (see Eq. (14.416)), making some simplifications and changing t into τ:

$$\Psi = \ln\left(\sqrt{\frac{n+2}{n}} \sqrt{\frac{\beta}{|\gamma|}}\right) + \ln\left(\frac{1 - C(-\tau)^{\widetilde{k}}}{1 + C(-\tau)^{\widetilde{k}}}\right). \tag{247}$$

Moreover what we really need is a derivative of Ψ.

$$\Psi' = \frac{d\Psi}{d\tau} = \frac{2C\widetilde{\kappa}(-\tau)^{\tilde{\kappa}-1}}{1 - C^2(-\tau)^{2\tilde{\kappa}}} \tag{248}$$

where

$$\widetilde{\kappa} = \frac{\overline{B}\sqrt{2n}}{H_1} \tag{248a}$$

$$C = e^{\overline{B}\sqrt{2n}t_0}\,(R_1H_1)^{\overline{B}\sqrt{2n}/H_1} = e^{\overline{B}\sqrt{2n}t_0}\,(R_1H_1)^{\tilde{\kappa}}\,. \tag{248b}$$

Now we proceed to a quintessence fluctuation equation (see Ref. [7]):

$$\delta\ddot{Q}_{\vec{k}} + 3H_1\delta\dot{Q}_{\vec{k}} + \left(c_s^2k^2 + R^2U''(Q)\right)\delta Q_{\vec{k}} = \frac{1}{2\overline{\beta}}h'_k\cdot\Psi'. \tag{249}$$

One gets

$$\begin{aligned}
&\delta\ddot{Q}_{\vec{k}} + 3H_1\delta\dot{Q}_{\vec{k}} + \left(c_s^2k^2 + R^2U''(Q)\right)\delta Q_{\vec{k}}\\
&\quad= \frac{C\widetilde{\kappa}}{\overline{\beta}}\left(A_{\vec{k}}\frac{(-\tau)^{\tilde{\kappa}-1}\sin(\tau|k|)}{1-C^2(-\tau)^{2\tilde{\kappa}}} + B_{\vec{k}}\frac{(-\tau)^{\tilde{\kappa}-1}\cos(\tau|k|)}{1-C^2(-\tau)^{2\tilde{\kappa}}}\right)\\
&\quad= \frac{C\widetilde{\kappa}}{\overline{\beta}}\,C_{\vec{k}}\,\frac{\sin(\tau|k|+\delta_{\vec{k}})}{1-C^2(-\tau)^{2\tilde{\kappa}}}\,(-\tau)^{2\tilde{\kappa}}.
\end{aligned} \tag{250}$$

$\overline{\beta}$ is a normalization constant in a definition of $Q = \frac{\Psi}{\overline{\beta}}$, $\frac{1}{\overline{\beta}^2} = \frac{8\pi|\overline{M}|}{m_{\rm pl}^2}$ (see (28)).

Let us consider $\delta Q_{\vec{k}}$ as a Fourier component of a q_0 field subject to the following initial conditions:

$$\frac{dq_{0\vec{k}}(0)}{d\tau} = 0 = q_{0\vec{k}}(0). \tag{251}$$

In this case one gets

$$\frac{d^2q_{0\vec{k}}}{d\tau^2} + 3H_1\frac{dq_{0\vec{k}}}{d\tau} + \left(c_s^2k^2 + \frac{m_0^2}{H_1^2\tau^2}\right)q_{0\vec{k}} = \frac{(-\tau)^{\tilde{\kappa}-1}C\widetilde{\kappa}}{\overline{\beta}\,(1-C^2(-\tau)^{2\tilde{\kappa}})}C_{\vec{k}}\sin\left(\tau|k|+\delta_{\vec{k}}\right), \tag{252}$$

c_s is a speed of sound in a quintessence, $C_{\vec{k}}$ and $\delta_{\vec{k}}$ are constants. Let us notice that $\tau = 0$ corresponds to $t \to \infty$. Thus in some sense we have asymptotic conditions for gravitational waves.

For further convenience it is good to change τ into $-\tau$. In this way we get

$$\frac{d^2q_{0\vec{k}}}{d\tau^2} - 3H_1\frac{dq_{0\vec{k}}}{d\tau} + \left(c_s^2k^2 + \frac{m_0^2}{H_1^2\tau^2}\right)q_{0\vec{k}} = -\frac{C\widetilde{\kappa}\tau^{\tilde{\kappa}-1}}{\overline{\beta}\,(1-C^2\tau^{2\tilde{\kappa}})}C_{\vec{k}}\sin\left(\tau|k|-\delta_{\vec{k}}\right), \tag{253}$$

$$\tau \in (0,+\infty), \quad \tau = \frac{1}{R_1H_1}e^{-H_1t}. \tag{254}$$

The fluctuations (the field q_0) of a quintessence Q are driven by gravitational waves and a quintessence field Q. It is easy to see that we are interested in solutions for small τ (large t). In this case one gets

$$\frac{d^2 q_{0\vec{k}}}{d\tau^2} - 3H_1 \frac{dq_{0\vec{k}}}{d\tau} + \frac{m_0^2}{H_1^2\tau^2} q_{0\vec{k}} = -\frac{C\widetilde{\kappa}\tau^{\tilde{\kappa}-1}}{\overline{\beta}} C_{\vec{k}} \sin(\tau|k| - \delta_{\vec{k}}). \tag{255}$$

Let us calculate a constant $\widetilde{\kappa}$. One gets

$$\begin{aligned}\widetilde{\kappa} = \frac{\overline{B}\sqrt{2n}}{H_1} &= \left(\frac{n}{n+2}\right)^{n/4} \left(\frac{|\gamma|}{\beta}\right)^{n/4} \frac{\sqrt{M}}{m_{\rm pl}} \cdot 4(n+2)|\gamma|^{1/2} \\ &= 8\sqrt{3\pi n}(n+2)\sqrt{M} \left(\frac{n}{n+2}\right)^{n/4} \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^{(n+2)/2} \frac{|\underline{\widetilde{P}}|^{1/2}}{\alpha_s^{(n+1)}} \left(\frac{|\underline{\widetilde{P}}|}{\widetilde{R}(\widetilde{\Gamma})}\right)^{n/2}.\end{aligned} \tag{256}$$

If we use our simplified model with $n = 14$, $\alpha_s^2 = \alpha_{\rm em}$, $m_{\tilde{A}} = m_{\rm EW}$, we get

$$\widetilde{\kappa} \cong 10^{92}. \tag{257}$$

Thus this is really a large number.

Moreover for small τ we have

$$q_{o\vec{k}}(\tau) \simeq 0, \quad \frac{dq_{0\vec{k}}}{d\tau} \simeq 0. \tag{258}$$

In this way we arrive to an equation

$$\frac{d^2 q_{0\vec{k}}}{d\tau^2} = -\frac{C\widetilde{\kappa}}{\overline{\beta}} \left(B_{\vec{k}} k \tau^{\tilde{\kappa}} + A_{\vec{k}} \tau^{\tilde{\kappa}-1}\right) \tag{259}$$

and finally

$$q_{0\vec{k}}(\tau) = -\frac{B_{\vec{k}}|k|C\widetilde{\kappa}}{\overline{\beta}(\widetilde{\kappa}+2)(\widetilde{\kappa}+1)} \tau^{\tilde{\kappa}+2} - \frac{A_{\vec{k}}C\widetilde{\kappa}}{\overline{\beta}(\widetilde{\kappa}+1)\widetilde{\kappa}} \tau^{\tilde{\kappa}+1} \tag{260}$$

or (taking under consideration that $\widetilde{\kappa}$ is very large)

$$q_{0\vec{k}}(\tau) = -\frac{C}{\overline{\beta}\widetilde{\kappa}} \left(B_{\vec{k}} k \tau^{\tilde{\kappa}} + A_{\vec{k}} \tau^{\tilde{\kappa}-1}\right) \tag{261}$$

$$\begin{aligned} q_{0\vec{k}}(\tau) = -\frac{C}{\overline{\beta}\widetilde{\kappa}} \Big(& A_{\vec{k}} (R_1 H_1)^{-\overline{B}\sqrt{2n}/H_1} e^{-\overline{B}\sqrt{2n}t} \\ & + k B_{\vec{k}} (R_1 H_1)^{-((\overline{B}\sqrt{2n}/H_1)-1)} e^{-\overline{B}\sqrt{2n}t} \cdot e^{H_1 t} \Big) \end{aligned} \tag{262}$$

$$q_0(t,\vec{r}) = \int q_{0\vec{k}}(t) e^{i\vec{k}\vec{r}}\, d\vec{k}. \tag{263}$$

One gets

$$q_0(t,\vec{r}) = -\frac{C}{\overline{\beta}\widetilde{\kappa}}\Bigg(g(\vec{r})\left(\frac{H_1}{R_1}\right)^{\overline{B}\sqrt{2n}/H_1} e^{-\overline{B}\sqrt{2n}t} + f(\vec{r})\left(\frac{H_1}{R_1}\right)^{(\overline{B}\sqrt{2n}/H_1)-1} e^{-2\overline{B}\sqrt{2n}t}\cdot e^{H_1 t}\Bigg) \tag{264}$$

where

$$f(\vec{r}) = \int A_{\vec{k}} e^{i\vec{k}\vec{r}}\, d^3\vec{k} \tag{265}$$

$$g(\vec{r}) = \int |k| B_{\vec{k}} e^{i\vec{k}\vec{r}}\, d^3\vec{k}. \tag{266}$$

$g(\vec{r})$ and $f(\vec{r})$ characterize a spatial dependence of gravitational waves background. For large t we have to do with $g(\vec{r})$, i.e.

$$q_0(t,\vec{r}) = -\frac{C}{\overline{\beta}\widetilde{\kappa}} g(\vec{r})(R_1H_1)^{-((\overline{B}\sqrt{2n}/H_1)-1)}\cdot e^{-\overline{B}\sqrt{2n}t}e^{H_1t}. \tag{267}$$

In this way one gets from an energy-momentum tensor for q_0

$$T_{\mu\nu} = \partial_\mu q_0\cdot\partial_\nu q_0 - \tfrac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha q_0\cdot\partial_\alpha q_0 + m_0^2 q_0^2\right) \tag{268}$$

$$\begin{aligned}
\rho_{q_0} &= T_{44} = \dot{q}_0^2 - \frac{1}{2}\left(\dot{q}_0^2 - |\overrightarrow{\nabla} q_0|^2 - m_0^2 q_0^2\right)\\
&= \frac{1}{2}\,q_0^2\left(\left(H_1 - \overline{B}\sqrt{2n}\right)^2 + |\overrightarrow{\nabla}\ln g(r)|^2 + m_0^2\right)\\
&= \frac{\rho_{\rm gr} 2n\overline{B}^2R_1^2\left(\left(H_1-\overline{B}\sqrt{2n}\right)^2 + |\overrightarrow{\nabla}\ln g(r)|^2 + m_0^2\right)}{\overline{\beta}^2 e^{2\overline{B}\sqrt{2n}t}}\, e^{2H_1t}e^{\overline{B}\sqrt{2n}t_0}
\end{aligned} \tag{269}$$

where

$$\rho_{\rm gr} = \tfrac{1}{2}g^2 \tag{270}$$

is an energy density of primordial gravitational waves.

For $\widetilde{\kappa} = \frac{\overline{B}\sqrt{2n}}{H_1}$ is a large number,

$$\overline{B}\sqrt{2n} \gg H_1 \tag{271}$$

and ρ_{q_0} is going to zero if $t\to\infty$. This density will be frozen (no dependence on time) at an end of the second de Sitter phase $t = t^{\rm II}_{\rm end}$, $t_0 = t^{\rm II}_{\rm initial}$. In this way one gets

$$\frac{\rho_{q_0}}{\rho_{\rm gr}} = \frac{2n\overline{B}^2R_1^2\left(\left(H_1-\overline{B}\sqrt{2n}\right)^2 + |\overrightarrow{\nabla}\ln g(r)|^2 + m_0^2\right)}{\overline{\beta}^2\exp\left(\frac{2\overline{B}\sqrt{2n}N_1}{H_1}\right)}\, e^{2N_1} \tag{272}$$

where N_1 is an amount of inflation during the second de Sitter phase. If $g(\vec{r}) = \text{const.}$, ρ_{q_0} is isotropic and homogeneous.

The function $g(\vec{r})$ can be written in a form

$$g(\vec{r}) = g_0 + \delta g(\vec{r}) \tag{273}$$

where $\delta g(\vec{r})$ is a small deviation and $g_0 = \text{const.}$ One gets

$$\overrightarrow{\nabla}\,(\ln g(\vec{r})) \simeq \frac{1}{g_0}\overrightarrow{\nabla}\delta g(\vec{r}). \tag{274}$$

In this way one writes

$$\rho_{q_0} = \rho^0_{q_0} + \delta\rho_{q_0} \tag{275}$$

where

$$\begin{aligned}
\rho^0_{q_0} &= \frac{2n\overline{B}^2 R_1^2 \rho_{\text{gr}}\left(\left(H_1 - \overline{B}\sqrt{2n}\right)^2 + m_0^2\right)}{\overline{\beta}^2 \exp\left(\frac{2\overline{B}\sqrt{2n}N_1}{H_1}\right)}\, e^{2N_1} \\
\delta\rho_{q_0} &= |\overrightarrow{\nabla} g(\vec{r})|^2\, \frac{2n\overline{B}^2 R_1^2 \rho_{\text{gr}}}{\overline{\beta}^2 \exp\left(\frac{2\overline{B}\sqrt{2n}N_1}{H_1}\right)}\, e^{2N_1}
\end{aligned} \tag{276}$$

According to our ideas the energy ρ_{q_0} should be stored as a gas of quintessence particles. The measurement of a low frequency speed of sound and a high frequency speed of sound can give us η and its spatial variation. In this way we get also a square length of a gradient of $\delta g(\vec{r})$. If we suppose that ρ_{gr} is stored as a gas of gravitons (massless particles with two polarization states) then one can write

$$p_{\text{gr}} = \rho_{\text{gr}}\left(\frac{T_{\text{gr}}}{\overline{T}_0}\right)^4 \tag{277}$$

(see Eq. (231)).

Probably we should also consider a gas of skewons. However, this is a different story. Taking into account (235) and (236) we should expect $\delta\rho_{q_0}$ as extremely small and we can neglect it in our theory. Moreover, ρ_{gr} is very important. Let us consider Eq. (253) for these values of τ for which the variable t is close to $t^{\text{I}}_{\text{end}}$ (the end of the first de Sitter phase or beginning of the second de Sitter phase). In this way one writes

$$\tau = \frac{1}{H_1 R_1}\, e^{-H_1 t^{\text{I}}_{\text{end}}} + \xi = \tau_0 + \xi, \quad 0 < \xi \ll 1. \tag{278}$$

Taking into account that ξ is very small we arrive to the following equation

$$\frac{d^2 q_{0\vec{k}}}{d\xi^2} - 3H_1\frac{dq_{0\vec{k}}}{d\xi} + \left(c_s^2 k^2 + \frac{m_0^2}{H_1^2\tau_0^2}\right) q_{0\vec{k}} = -\frac{C_{\vec{k}}\tau_0}{2\xi\overline{\beta}}\sin\left(\xi|k| + \overline{\delta}_{\vec{k}}\right) \tag{279}$$

where

$$\overline{\delta}_{\vec{k}} = \tau_0|k| - \delta_{\vec{k}}\,. \tag{280}$$

We have also initial conditions

$$\frac{dq_{0\vec{k}}}{d\xi}(0) = q_{0\vec{k}}(0) = 0. \tag{281}$$

Eq. (279) can be solved by a Laplace transform method in the case of $\overline{\delta}_{\vec{k}} = 0$.

Let

$$c_s^2k^2 + \frac{m_0^2}{H_1^2\tau_0^2} = b^2. \tag{282}$$

One gets

$$s^2\overline{q}_{0\vec{k}}(s) - 3H_1 s\overline{q}_{0\vec{k}}(s) + b^2\overline{q}_{0\vec{k}}(s) = -\frac{C_{\vec{k}}\tau_0}{2\overline{\beta}}\,\mathrm{arctg}\left(\frac{|k|}{s}\right) \tag{283}$$

$$\overline{q}_{0\vec{k}}(s) = -\frac{C_{\vec{k}}\tau_0}{2\overline{\beta}}\,\frac{\mathrm{arctg}\left(\frac{|k|}{s}\right)}{s^2 - 3H_1 s + b^2} \tag{284}$$

and

$$q_{0\vec{k}}(\xi) = -\frac{C_{\vec{k}}\tau_0}{2\overline{\beta}}\int\limits_0^\infty \frac{\mathrm{arctg}\left(\frac{|k|}{s}\right)e^{s\xi}}{s^2 - 3H_1 s + b^2}\,ds. \tag{285}$$

However, we cannot get any compact form of this solution. Moreover for we are looking for a solution around zero, for $\overline{\delta}_{\vec{k}} = 0$ we get

$$-\frac{C_{\vec{k}}\tau_0}{2\xi}\sin(\xi|k|) \simeq -\frac{C_{\vec{k}}\tau_0|k|}{2}\,. \tag{286}$$

In this case the Laplace transform method is successful:

$$q_{0\vec{k}}(\xi) = -\frac{C_{\vec{k}}\tau_0|k|}{2\overline{\beta}}\int\limits_0^\infty \frac{e^{s\xi}}{s(s^2 - 3H_1 s + b^2)}\,ds \tag{287}$$

$$q_{0\vec{k}}(\xi) = -\frac{C_{\vec{k}}\tau_0|k|}{2\overline{\beta}}\left(\frac{1}{b^2} + \frac{e^{s_1\xi}}{s_1(s_1 - s_2)} + \frac{e^{s_2\xi}}{s_2(s_2 - s_1)}\right) \tag{287a}$$

where s_1 and s_2 are roots of the polynomial

$$s^2 - 3H_1 s + b^2 = 0. \tag{288}$$

Moreover ξ is small. Thus we write

$$e^{s_i\xi} \cong 1 + s_i\xi, \quad i = 1, 2, \tag{289}$$

and get

$$q_{0\vec{k}}(\xi) = -\frac{C_{\vec{k}}\tau_0|k|}{2b^2\overline{\beta}}\left(1+\frac{3H_1}{\sqrt{\Delta}}\right) \tag{290}$$

if

$$\Delta = 9H_1^2 - 4b^2 = 9H_1^2 - 4\Big(c_s^2k^2 + \frac{m_0^2}{H_1^2\tau_0^2}\Big) > 0. \tag{291}$$

If $\Delta < 0$, one gets

$$q_{0\vec{k}}(\xi) = -\frac{C_{\vec{k}}\tau_0|k|}{2b^2\overline{\beta}}\left(1+\exp(\tfrac{3}{2}H_1\xi)\left(\frac{3H_1}{\omega}\sin(\omega\xi)-\cos(\omega\xi)\right)\right) \tag{292}$$

where

$$4\omega^2 = -\Delta = 4\Big(c_s^2k^2 + \frac{m_0^2}{H_1^2\tau_0^2}\Big) - 9H_1^2 > 0. \tag{293}$$

For small ξ one gets

$$q_{0\vec{k}}(\xi) = -\frac{3C_{\vec{k}}\tau_0|k|H_1}{4b^2\overline{\beta}}\,\xi, \tag{294}$$

the dependence on t is as follows:

$$\xi = \frac{-1}{R_1H_1}\big(e^{-H_1t} - e^{-H_1t^{\mathrm{I}}_{\mathrm{end}}}\big). \tag{295}$$

Moreover t is close to $t^{\mathrm{I}}_{\mathrm{end}}$ and we get

$$\xi \simeq \frac{e^{-H_1t^{\mathrm{I}}_{\mathrm{end}}}}{R_1}(t - t^{\mathrm{I}}_{\mathrm{end}}) \tag{296}$$

for t very close to $t^{\mathrm{I}}_{\mathrm{end}}$.

In this way we get fluctuations of a quintessence in the moment of the first order phase transition of the first de Sitter phase to the second de Sitter phase.

Finally one gets

$$\begin{aligned} q_{0\vec{k}}(t) = &-\frac{C_{\vec{k}}|k|e^{-H_1t^{\mathrm{I}}_{\mathrm{end}}}}{2H_1R_1\left(c_s^2|k|^2 + R_1^2m_0^2e^{2H_1t^{\mathrm{I}}_{\mathrm{end}}}\right)\overline{\beta}} \\ &\times\left(1+\frac{3H_1}{\sqrt{9H_1^2 - 4\big(c_s^2|k|^2 + R_1^2m_0^2e^{2H_1t^{\mathrm{I}}_{\mathrm{end}}}\big)}}\right) \end{aligned} \tag{297}$$

for

$$|k| < \frac{1}{2c_s}\sqrt{9H_1^2 - 4R_1^2e^{2H_1t^{\mathrm{I}}_{\mathrm{end}}}m_0^2} = k_0 \tag{298}$$

with a condition

$$3H_1 > 2R_1 e^{H_1 t^{\mathrm{I}}_{\mathrm{end}}} m_0 \tag{299}$$

and

$$q_{0\vec{k}}(t) = -\frac{3C_{\vec{k}} e^{-2H_1 t^{\mathrm{I}}_{\mathrm{end}}}(t - t^{\mathrm{I}}_{\mathrm{end}})}{4R_1^2 \left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)\overline{\beta}}\,. \tag{300}$$

One finds

$$\begin{aligned} q_0(\vec{r},t) = -\frac{e^{-H_1 t^{\mathrm{I}}_{\mathrm{end}}}}{2\overline{\beta} R_1} & \left(\frac{1}{H_1} \int\limits_{|k|<k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}|k|}{\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)} \right. \\ & \times \left(1 + \frac{3H_1}{\sqrt{9H_1^2 - 4\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)}} \right) \\ & \left. + \frac{3(t - t^{\mathrm{I}}_{\mathrm{end}})}{2R_1} e^{-H_1 t^{\mathrm{I}}_{\mathrm{end}}} \int\limits_{|k|>k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}}{\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)} \right) \end{aligned} \tag{301}$$

Equation (301) gives us space-time fluctuations of a quintessence field after a phase transition from the first to the second de Sitter phase. Let us notice that (297) is singular for $k \to k_0$.

In an approach developed here we consider Eq. (250) with zero initial conditions in various approximations. Moreover there is a different approach. In this approach we write a solution to Eq. (250)

$$q_{0\vec{k}}(t) = \widetilde{q}_{0\vec{k}}(t) + \overline{q}_{0\vec{k}}(t) \tag{302}$$

where $\widetilde{q}_{0\vec{k}}(t)$ is a solution of a homogeneous equation (a general integral) and $\overline{q}_{0\vec{k}}(t)$ is a special solution to inhomogeneous equation. This different approach consists in taking $\widetilde{q}_{0\vec{k}}(t) \equiv 0$ and considering only $\overline{q}_{0\vec{k}}(t)$. In this case we get nonzero initial conditions for quintessence fluctuations. However, this approach is not unambigous. Our approach is unambigous. The zero initial conditions are in some sense distinguished.

Let us calculate an energy density of this scalar field. In order to do this let us write $q_0(\vec{r},t)$ in a form

$$q_0(\vec{r},t) = A(\vec{r}) + B(\vec{r})(t^{\mathrm{I}}_{\mathrm{end}} - t) \tag{303}$$

where

$$A(\vec{r}) = -\frac{e^{-H_1 t^{\mathrm{I}}_{\mathrm{end}}}}{2\overline{\beta} R_1 H_1} \int\limits_{|k|<k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}|k|}{\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)}$$

$$\times \left(1 + \frac{3H_1}{\sqrt{9H_1^2 - 4\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)}}\right) \tag{304}$$

$$B(\vec{r}) = -\frac{3e^{-H_1 t^{\mathrm{I}}_{\mathrm{end}}}}{4\overline{\beta} R_1^2} \int\limits_{|k|>k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}}{\left(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}}\right)} . \tag{305}$$

One gets from the Eq. (268)

$$\rho_{q_0} = T_{44} = \frac{1}{2}\left(C(\vec{r}) + D(\vec{r})(t^{\mathrm{I}}_{\mathrm{end}} - t) + E(\vec{r})(t^{\mathrm{I}}_{\mathrm{end}} - t)^2\right) \tag{306}$$

where

$$C(\vec{r}) = B^2(\vec{r}) + m_0^2 A^2(\vec{r}) + \left|\overrightarrow{\nabla}(A(\vec{r}))\right|^2 \tag{307}$$

$$D(\vec{r}) = 2\left(\overrightarrow{\nabla}(A(\vec{r})) \cdot \overrightarrow{\nabla}(B(\vec{r})) + m_0^2 A(\vec{r}) B(\vec{r})\right) \tag{308}$$

$$E(\vec{r}) = \left(\left|\overrightarrow{\nabla}(B(\vec{r}))\right|^2 + m_0^2 B^2(\vec{r})\right) . \tag{309}$$

All these formulae are satisfied for t close to $t^{\mathrm{I}}_{\mathrm{end}}$. In general ρ_{q_0} is anisotropic and inhomogeneous. For $t^{\mathrm{I}}_{\mathrm{end}} - t$ is very small we get

$$\rho_{q_0} = \frac{1}{2}\left(C(\vec{r}) + D(\vec{r})(t^{\mathrm{I}}_{\mathrm{end}} - t)\right) . \tag{310}$$

It is interesting to calculate a time $t(\vec{r})$ for which ρ_{q_0} is zero ($t(\vec{r})$ depends on a space point). One gets (neglecting gradients of $A(\vec{r})$ and $B(\vec{r})$)

$$t(\vec{r}) = t^{\mathrm{I}}_{\mathrm{end}} + \frac{1}{2m_0^2}\left(\overline{f}(\vec{r}) + \frac{m_0^2}{\overline{f}(\vec{r})}\right) \tag{311}$$

where

$$\overline{f}(\vec{r}) = \frac{3H_1}{2R_1} \cdot \frac{\int_{|k|>k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}}{(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}})}}{\int_{|k|<k_0} d^3\vec{k}\, e^{i\vec{k}\vec{r}} \frac{C_{\vec{k}}|k|}{(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}})}\left(1 + \frac{3H_1}{\sqrt{9H_1^2 - 4(c_s^2 k^2 + R_1^2 m_0^2 e^{2H_1 t^{\mathrm{I}}_{\mathrm{end}}})}}\right)} . \tag{312}$$

The conclusion from these calculations is that after a very short time an energy density of quintessence fluctuations is going to zero (almost zero) and probably is frozen on a very low level. A time of this process depends on a space point. Moreover, after sufficiently long time (which is really very short) we get a homogeneity and isotropy, as far as our rough calculations advise us. Let us come back to Eq. (253). Our solutions obtained here are for large t and t close to $t^{\mathrm{I}}_{\mathrm{end}}$. One can try to match them. It seems reasonable to suppose that $\rho^0_{q_0}$ (see Eq. (276)) is equal to this very low level energy density, mentioned above.

The quintessence field $q_0(\vec{r},t)$ considered above can be a source of fluctuations of an effective gravitational constant G_{eff}.

$$G_{\mathrm{eff}} = G_N \exp\left(-(n+2)\overline{\beta} q_0(\vec{r},t)\right) \tag{313}$$

(see Eq. (28)). If we use Eq. (267), we get

$$G_{\mathrm{eff}} = G_N \exp\left(-\frac{\sqrt{2}\,(n+2)\sqrt{\rho_{q_0}}\,\overline{\beta}}{\sqrt{\left((H_1-\overline{B}\sqrt{2n})^2+\left|\overrightarrow{\nabla}\ln g(\vec{r})\right|^2+m_0^2\right)}}\right) \tag{314}$$

and we connect contemporary fluctuations of an effective gravitational constant to an energy density of quintessence fluctuations. Taking a quintessence field from Eq. (303) we get

$$G_{\mathrm{eff}} = G_N \exp\left(-(n+2)\overline{\beta}\left(A(\vec{r})+B(\vec{r})(t^{\mathrm{I}}_{\mathrm{end}}-t)\right)\right). \tag{315}$$

In this way an effective gravitational constant depends on a speed of a sound in a quintessence.

We can consider some applications of a quintessence in astrophysics. First of all we can develop a formalism concerning a macroscopic flow of a quintessence gas (i.e., a relativistic hydrodynamics of a quintessence). For a quintessence field has an influence on an effective gravitational constant, such a flow can disturb gravitational interactions between galaxies or even stars. In this approach we should add some assumptions, e.g. that a quintessence energy density is an energy density of quintessence field in statistical field theory. However, this is beyond a scope of this work.

Secondly, it is interesting to consider a Bose-Einstein condensation of quintessence particles. They are massive scalar particles. Under some assumptions (beyond this theory) we can find a wave function of a condensat. This wave function can be considered a field of a quintessence and afterwards enters the formula for G_{eff}. In this way Einstein-Bose condensation can influence effective gravitational interactions between galaxies and stars. This is also beyond a scope of the work.

As a typical astrophysical application we can consider a problem of N bodies with the quintessence field. In this problem N points interact gravitationally with an effective gravitational constant depending on a distance. The simplest case is of course a two bodies problem. However, this problem cannot be solved analytically. It can be considered numerically using some codes for the N bodies problem. This is also

beyond a scope of our work. Moreover, some numerical solutions can be applied for galaxies movement in a cluster of galaxies.

In all of those problems we should consider quintessence fluctuations developed here as a source of quintessence gas and Bose-Einstein condensat. In the third problem we should also consider an interaction of a quintessence field with mass points.

It is interesting to consider a continuous distribution of a matter (a dust) under gravitational interaction and a quintessence. In this way we consider hydrodynamic equation coupled to a quintessence field in a Newtonian physics limit (i.e. selfgravitating system with G_{eff} depending on a quintessence).

Conclusions

In the paper we consider a mass of a quintessence particle, a speed of sound in a quintessence and several solutions to quintessence equations coming to the interesting behaviour of an effective gravitational constant. Some of issues of our theory are considered in self-interacting Brans-Dike theory [8].

The fraction η can be connected with the part of an energy density of quintessence field Q in such a way that it is a fraction of fluctuation energy density around an equilibrium Q_0. In this way the q_0 field which can evolve due to an evolution of Q and due to fluctuations of primordial gravitational waves will be a source of a gas of thermalized particles. It seems that an approach with a boson equation of state is more appropriate to consider.

References

[1] Kalinowski M. W., *A short note on a total amount of inflation*, arXiv: hep-th/0307299v2.

[2] Kalinowski M. W., *Nonsymmetric Fields Theory and its Applications*, World Scientific, Singapore, New Jersey, London, Hong Kong 1990.
Kalinowski M. W., *Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory in the electromagnetic case*, Int. Journal of Theor. Phys. **31**. p. 611 (1992).
Kalinowski M. W., *Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory in a general nonabelian case*, Int. Journal of Theor. Phys. **30**, p. 281 (1991).
Kalinowski M. W., *Can we get confinement from extra dimensions*, in: Physics of Elementary Interactions (ed. Z. Ajduk, S. Pokorski, A. K. Wróblewski), World Scientific, Singapore, New Jersey, London, Hong Kong 1991.
Kalinowski M. W., *Scalar fields in the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory*, arXiv: hep-th/0307242.

[3] Kalinowski M. W., *A warp factor in the nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory*, arXiv: hep-th/0306157v2.

[4] Kalinowski M. W., *Dynamics of Higgs' field and a quintessence in the nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory*, arXiv: hep-th/0306241v1.

[5] Emden R., *Gaskugeln: Anwendungen der mechanischer Wärme*, B. G. Teubner, Leipzig und Berlin 1907.
Chandrasekhar S., *Introduction to the Study of Stellar Structure*, New York, Dover 1939.

[6] Lemke H., *Über die Differentialgleichungen, welche den Gleichgewichtszustand eines gasförmigen Himmelskörpers bestimmen, dessen Teile gegeneinander nach dem Newtonschen Gesetze gravitieren*, Journal für Mathematik, Bd. 142, Heft 2, S. 118 (1913).

[7] Steinhardt P. J., *The quintessential introduction to dark energy*, Lecture given at The search of dark matter and dark energy in the Universe, The Royal Society, January 22–23, 2003.
DeDeo S., Caldwell R. R., Steinhardt P. J., *Effects of the sound speed of quintessence on the microwave background and large scale structure*, arXiv: astro-ph/0301284v2.
Erickson J. K., Caldwell R. R., Steinhardt P. J., Armendriz-Picon C., Mukhanov V., *Measuring the speed of sound of quintessence*, arXiv: astro-ph/0112438v1.
Rahul D., Caldwell R. R., Steinhardt P. J., *Sensitivity of the cosmic microwave background anisotropy to initial conditions in quintessence cosmology*, arXiv: astro-ph/0206372v2.
Caldwell R. R., Steinhardt P. J., *The imprint of gravitational waves in models dominated by a dynamical cosmic scalar field*, arXiv: astro-ph/9710062v1.
Steinhardt P. J., *Quintessential cosmology and cosmic acceleration*, see Steinhardt's home WWW page.

[8] Chakraborty S., Chakraborty N. C., Debnath U., *A quintessence problem in self-interacting Brans-Dicke theory*, arXiv: gr-gc/0305103v1.
Chakraborty S., Chakraborty N. C., Debnath U., *Quintessence problem and Brans-Dicke theory*, arXiv: gr-qc/0306040v1.

[9] Liddle A. R., Lyth D. H., *Cosmological Inflation and Large-Scale Structure*, Cambridge Univ. Press, Cambridge 2000.